Some Properties of matrix operations

$A(B+C) = AB+AC$

$(A+B)C = AC+BC$

$A(BC) = (AB)C$

$AB \neq BA$ generally

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

$(A^T)^T = A$

Invertible matrix thm (v. 1)

Let A be $n\times n$ matrix. The following statements are equivalent:

a) A is invertible

b) $A\vec{x}=\vec{b}$ has a unique solution for every $\vec{b}\in\mathbb{R}^n$

c) $A\vec{x}=\vec{0}$ has only the trivial solution

d) RREF of A is I_n

a) $\Rightarrow$ b) : for A invertible, the unique sol. of $A\vec{x}=\vec{b}$ is $\vec{x}=A^{-1}\vec{b}$.

Ex: $\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\vec{b}} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Block multiplication - Poole pp. 148-149.

[Poole 2.3]

Linear independence

def A set of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$ is <u>linearly dependent</u> if there are scalars $c_1, \dots, c_k$ (not all zero) s.t. $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ (linear dependence relation)

- a set of vectors that is not lin. dependent is called <u>linearly independent</u>

Thm a set of vectors is lin. dependent iff one of the vectors can be written as a lin. comb. of the others.

[Idea: ($\Rightarrow$): $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ lin. dep. relation assume $c_1 \neq 0$. $\to$ divide by c_1

Then: $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \dots - \frac{c_k}{c_1}\vec{v}_k$

($\Leftarrow$): say, $\vec{v}_1 = d_2\vec{v}_2 + \dots + d_k\vec{v}_k$. Then: $\vec{v}_1 - d_2\vec{v}_2 - \dots - d_k\vec{v}_k = \vec{0}$ lin. dep. rel.]

Ex. The set $\vec{0}, \vec{v}_2, \dots, \vec{v}_k$ is lin. dep.: $1\cdot\vec{0} + 0\cdot\vec{v}_2 + \dots + 0\cdot\vec{v}_k = \vec{0}$

- a single vector $\{\vec{v}\}$ is lin. indep. iff $\vec{v} \neq \vec{0}$.
- a set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is lin. dependent iff one is a multiple of the other.

Ex: $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ lin. indep., $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}\right\}$ lin. dep.

Ex: $\left\{\underbrace{\begin{bmatrix}1\\2\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\1\\-1\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}1\\4\\2\end{bmatrix}}_{\vec{v}_3}\right\}$ - lin. dep. or not?

Sol: $c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Aug. Mat $\left[\begin{array}{ccc|c}1&1&1&0\\2&1&4&0\\0&-1&2&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{array}\right]$

$c_1 \quad c_2 \quad c_3 = s$

free variable

$c_1 = -3s$
$c_2 = 2s$
$c_3 = s$

∞-many solutions, in part. nontriv. solutions

$\Rightarrow$ the set is <u>lin. dep.</u>

e.g. setting $s = 1$: $c_1 = -3, c_2 = 2, c_3 = 1$

$\Rightarrow \boxed{-3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}}$ lin. dep. relation.

Thm: a set of k vectors in $\mathbb{R}^n$ is always lin. dep. if $k > n$.

Subspaces (Poole 3.5)

def A <u>subspace</u> of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

(a) The vector $\vec{0}$ is in S

(b) If $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$ ("S is closed under addition")

(c) If $\vec{u} \in S$, $\underset{\text{a scalar}}{c \in \mathbb{R}}$, then $c\vec{u} \in S$ ("S is closed under scalar multiplication")

- (b) + (c) $\Rightarrow$ if $\vec{v}_1, \dots, \vec{v}_k \in S$, $c_1, \dots, c_k$ scalars, then $c_1\vec{v}_1 + \dots + c_k\vec{v}_k \in S$

<u>Ex:</u> let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. Then $S = \text{span}(\vec{v}_1, \vec{v}_2)$ is a subspace of $\mathbb{R}^n$.

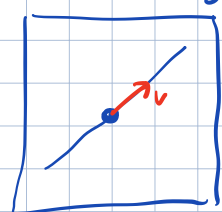
check: (a) $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \in S$ ✓

(b) $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$, $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2$ $\Rightarrow \vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 \in S$ ✓

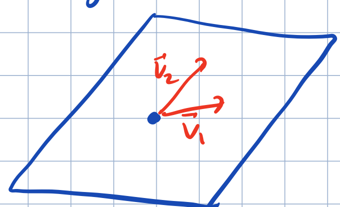
(c) $c\vec{u} = (cc_1)\vec{v}_1 + (cc_2)\vec{v}_2 \in S$ ✓

<u>Generally:</u> for $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, $S = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$
"subspace spanned by $\vec{v}_1, \dots, \vec{v}_k$"

<u>Remark:</u> • For $\vec{v} \neq \vec{0}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\text{span}(\vec{v})$ - a line through the origin.



- For $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ lin. indep., $\text{span}(\vec{v}_1, \vec{v}_2)$ is a plane through $\vec{0}$

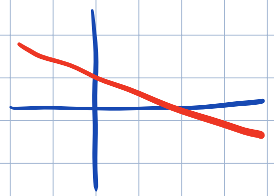


- For $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ lin. indep., $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3$ space.

<u>Ex:</u> $S = \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Also, $S = \{\vec{0}\}$ is a subspace - "zero subspace"

Ex: A line L in $\mathbb{R}^2$ not through the origin is not a subspace

$(\vec{0} \notin L \Rightarrow$ axiom (a) fails $)$



Ex: $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{matrix} x = 3y \\ z = -2y \end{matrix} \right\} = \left\{ \begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right\}$ - subspace in $\mathbb{R}^3$

Ex: $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + z = 1 \right\}$ is <u>not a subspace</u> (does not contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

Ex: $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| y = x^2 \right\}$ is <u>not a subspace</u> $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S \text{ but } \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin S \right)$

Subspaces associated with a matrix

def Let A be an $m \times n$ matrix

1. The row space of A is the subspace row(A) of $\mathbb{R}^n$ spanned by rows of A
2. The column space of A is the subspace col(A) of $\mathbb{R}^m$ spanned by columns of A

Remark: $\text{col}(A) = \{ \text{vectors of the form } A\vec{x},\ \vec{x} \in \mathbb{R}^n \}$

Ex: $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$

a) is $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in col(A) ?

b) is $\vec{w} = [4 \ \ 5]$ in row(A)?

c) describe row(A) and col(A)

Sol a) $\vec{b} \in \text{col}(A)$ iff lin. sys. $A\vec{x} = \vec{b}$ is consistent

Aug. Mat. $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ consistent $\Rightarrow \vec{b} \in \text{col}(A)$

b) $\vec{w} \in \text{row}(A)$ iff $\begin{bmatrix} A \\ \hline \vec{w} \end{bmatrix} \xrightarrow[\substack{\text{using only operations} \\ R_i + cR_j,\ i > j}]{\text{row reduction}} \begin{bmatrix} A' \\ \hline \vec{0} \end{bmatrix}$

some matrix

zero row

- since elem. row operations create lin. comb. of rows

$$\left[\frac{A}{\vec{w}}\right] = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right] \xrightarrow[R_4-4R_1]{R_3-3R_1} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{array}\right] \xrightarrow{R_4-9R_2} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow \vec{w} \in \text{row}(A)$$

c) similarly

$$\left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline x & y \end{array}\right] \to \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow \text{ any } [x \ y] \text{ is in row}(A),$$

so $\text{row}(A) = \mathbb{R}^2$

$$\text{col}(A): \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & z-3x \end{array}\right] \text{ consistent iff } z-3x=0$$

So, $\text{col}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, z-3x=0 \right\}$

Thm If matrix A is row equivalent to B then $\boxed{\text{row}(A) = \text{row}(B)}$

def Let A be an $m \times n$ matrix. The null space of A is the set of all solutions of the homogeneous eq. $A\vec{x} = \vec{0}$. It is denoted $\text{null}(A)$.

- $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ is $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ in $\text{null}(A)$?

Sol: $A\vec{v} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ so $\vec{v} \in \text{null}(A)$.