LAST TIME

- $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ (set) is a <u>basis</u> for a subspace S in $\mathbb{R}^n$ if
  1) $S = \text{span}(B)$
  2) B is lin. indep.

- basis for row(A) = { nonzero rows of REF of A }

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{[1\ 0\ 1\ 0\ -1], [0\ 1\ 2\ 0\ 3], [0\ 0\ 0\ 1\ 4]\}$ - basis for row(A)

- basis for col(A) = { pivotal columns of A }

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$ - basis for col(A)

- null(A) = { solutions of $A\vec{x} = \vec{0}$ } $= \left\{ s \underbrace{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\vec{v}} \right\} = \text{Span}(\underbrace{\vec{u}, \vec{v}}_{\text{lin. indep.}})$

solve by row reduction

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$ - basis for null(A).

- To find a basis for null(A), solve $\overset{\text{RREF of A}}{R}\vec{x} = \vec{0}$ for leading vars in terms of free vars. Write the general sol. as a lin. comb. of $\underbrace{k}_{\#\text{ free vars}}$ vectors $\vec{v}_1, \dots, \vec{v}_k$ with parameters = free vars. These $\vec{v}_1, \dots, \vec{v}_k$ form a basis for null(A).

Remark: Another way to construct a basis for col(A): construct a basis for row(A^T) and transpose row basis vectors back into columns.

Ex:
$$A^T = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ 1 & 1 & -2 & 1 \\ 6 & -1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \{[1\ 0\ 0\ 1], [0\ 1\ 0\ 6], [0\ 0\ 1\ 3]\}$$
- basis for row(A^T)

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$
- basis for col(A) (different from the one we constructed before!)

Dimension

Basis theorem Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

def If S is a subspace of $\mathbb{R}^n$, the number of vectors in a basis for S is called the dimension of S. Notation: $\dim S$.

Ex: $\mathbb{R}^n$ has a basis $\{\underbrace{\vec{e}_1, \dots, \vec{e}_n}_{n}\}$ $\Rightarrow$ $\dim \mathbb{R}^n = n$.

For A $m \times n$ matrix,

		in Example above
$\dim \text{row}(A) = \#$ nonzero rows in REF of A	$= \#$pivots $= \text{rank}(A)$	3 — rank(A)
$\dim \text{col}(A) = \#$ pivotal columns	$= \#$pivots $= \text{rank}(A)$	3 — rank(A)
$\dim \text{null}(A) = \#$ free vars in $A\vec{x} = \vec{0}$		2 — nullity(A)

def Nullity of A is the dimension of null(A).

Thm: For any matrix A, $\dim \text{row}(A) = \dim \text{col}(A)$ $(= \text{rank}(A))$

So, rank(A) = # lin. indep. rows in A $\underset{\text{surprisingly!}}{=}$ # lin. indep. columns in A

Corollary: rank(A) = rank(A^T)

(rank(A) = dim row(A) = dim col(A^T) = rank(A^T))

Rank thm: $\boxed{\underbrace{\text{rank}(A)}_{\#\text{ pivotal columns in } A} + \underbrace{\text{nullity}(A)}_{\#\text{ non-pivotal columns}} = n}$

Ex: for 4×5 matrix A above, dim row(A) = $\underset{\text{rank}}{3}$ = dim col(A),

dim null(A) = 2 ← nullity

$\underset{\text{rank}}{3} + \underset{\text{nullity}}{2} = \underset{\#\text{ columns}}{5}$

- for A $m\times n$ matrix, $A\vec{x}=\vec{b}$ is consistent for any $\vec{b}\in\mathbb{R}^m$ iff REF of A has a pivot in every row.

Fundamental Thm of invertible matrices, v.2

Let A be an $n\times n$ matrix. The following statements are equivalent:

(a) A is invertible

(b) $A\vec{x}=\vec{b}$ has a unique solution for any $\vec{b}$ in $\mathbb{R}^n$

(c) $A\vec{x}=\vec{0}$ has only the trivial solution

(d) RREF of A is I_n

(e) A is a product of elementary matrices ← we didn't discuss this

(f) rank$(A) = n$

(g) nullity$(A) = 0$

(h) column vectors of A are lin. indep.

(i) column vectors of A span $\mathbb{R}^n$

(j) column vectors of A form a basis for $\mathbb{R}^n$

(k) row vectors of A are lin. indep.

(l) row vectors of A span $\mathbb{R}^n$

(m) row vectors of A form a basis for $\mathbb{R}^n$

Ex: show that vectors $\underset{\vec{u}}{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix}4\\9\\7\end{bmatrix}}$ form a basis for $\mathbb{R}^3$

Sol: $A = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{bmatrix}1 & -1 & 4\\ 2 & 0 & 9\\ 3 & 1 & 7\end{bmatrix}$ ((f) $\Leftrightarrow$ (j) implies $\{\vec{u},\vec{v},\vec{w}\}$ is a basis iff $\operatorname{rank}(A)=3$)

$\rightarrow \underset{\text{REF}}{\begin{bmatrix}1 & -1 & 4\\ 0 & 2 & 1\\ 0 & 0 & 7\end{bmatrix}}$ $\operatorname{rank}(A) = 3$ ✓

Thm Let A be an $m\times n$ matrix, then

a) $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$

b) the $n\times n$ matrix A^TA is invertible iff $\operatorname{rank}(A)=n$.

(a) $\operatorname{null}(A^TA) = \operatorname{null}(A)$ – because $A\vec{x}=\vec{0} \Rightarrow A^TA\vec{x}=\vec{0}$, so $\operatorname{null}(A) \subset \operatorname{null}(A^TA)$

and $A^TA\vec{x}=\vec{0} \Rightarrow x^TA^TA\vec{x}=0 \Rightarrow$
$\Rightarrow (A\vec{x})\cdot(A\vec{x})=0 \Rightarrow A\vec{x}=0$

Thus: $\operatorname{nullity}(A^TA) = \operatorname{nullity}(A)$

$\underset{\text{rank-nullity Thm}}{\Rightarrow}$ $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$

(b) A^TA invertible $\underset{(a)\Leftrightarrow(f)}{\Leftrightarrow}$ $\operatorname{rank}(A^TA)=n \Leftrightarrow \operatorname{rank}(A)=n$.

Ex: $\overset{1\times 2}{A} = [1\ \ 2]$ rank $=1$ $\quad A^TA = \begin{bmatrix}1\\2\end{bmatrix}[1\ \ 2] = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ rank $=1$ non-invertible!

Ex: (for rank-nullity) A is 2×5. What are the possible values of $\operatorname{nullity}(A)$?

Sol: rank A can be $0,1,2$ $\Rightarrow$ $\operatorname{nullity}(A) = 5 - \operatorname{rank}(A)$ can be $\underline{\underline{3,4,5}}$.

Ex: find a basis for $S = \text{span}(\vec{u}, \vec{v}, \vec{w})$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 5 \end{bmatrix}$

Sol:

$S = \text{col}(A)$, $\quad A = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 5 \end{bmatrix} \\ \vec{u} \quad \vec{v} \quad \vec{w} \end{array} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

pivots

REF

$\Rightarrow \{\vec{u}, \vec{v}\}$ - basis for S.
= pivotal columns of A

Proof of Basis thm

Lemma Let $\vec{v}_1, \dots, \vec{v}_k$ be a set of vectors in $\mathbb{R}^n$ and $\vec{w}_1, \dots, \vec{w}_\ell$ a set of vectors in $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$. Then, if $\ell > k$, the set $\vec{w}_1, \dots, \vec{w}_\ell$ is lin. dependent.

⌈ We have $\vec{w}_i = \underbrace{a_{i1}\vec{v}_1 + \dots + a_{ik}\vec{v}_k}_{\text{some coefficients}}$, $i = 1 \dots \ell$. Let $\vec{a}_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{ik} \end{bmatrix}$.

Matrix $A = [\vec{a}_1 \dots \vec{a}_\ell]$ has # columns > # rows $\Rightarrow$ columns of A are lin. dep.

$\Rightarrow \quad c_1\vec{a}_1 + \dots + c_\ell\vec{a}_\ell = \vec{0}$ some lin. dep. rel.

$\Rightarrow \quad c_1\vec{w}_1 + \dots + c_\ell\vec{w}_\ell = \vec{0}$ - a lin. dep. rel. on $\vec{w}_i$'s ⌋

⌈ Argument for basis theorem: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$, $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ two bases for S.
Assume $\ell > k$. Then by Lemma, $\mathcal{C}$ is lin. dep. - contradiction! ⌋