Back to linear transformations (for a moment)

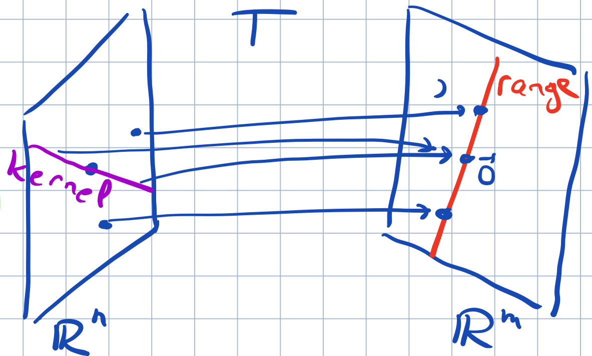
Let $\underset{\substack{\uparrow \\ \text{domain}}}{T: \mathbb{R}^n} \to \underset{\substack{\uparrow \\ \text{codomain}}}{\mathbb{R}^m}$ be a linear transformation.

"Range" of T = the set of all images $T(\vec{x})$ for any $\vec{x} \in \mathbb{R}^n$

"Kernel" of T = $\{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = 0 \}$ - all vectors that are mapped to zero by T.

range(T) is a subspace of $\mathbb{R}^m$

ker(T) is a subspace of $\mathbb{R}^n$



Ex: For $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$ a matrix transformation,

range(T) = col(A)

and ker(T) = null(A)

- $T: \mathbb{R}^n \to \mathbb{R}^m$ is "onto" if range(T) = codomain(T) $(= \mathbb{R}^m)$
- T is "one-to-one" if each $\vec{b} \in$ range(T) is the image of exactly one vector in the domain.

(equivalently, ker$(T) = \{\vec{0}\}$ - zero subspace of $\mathbb{R}^n$)

Ex: A matrix transf $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$ is onto if $A\vec{x} = \vec{b}$ consistent $\forall \vec{b}$

<=> if A has a pivot in each row

<=> if $\boxed{\text{rank}(A) = m}$

T is 1-1 <=> null$(A) = \{\vec{0}\}$

<=> nullity$(A) = 0$.

- we can consider linear transformations between subspaces,

$$T: \underset{\substack{\cap \\ \mathbb{R}^n}}{S} \to \underset{\substack{\cap \\ \mathbb{R}^m}}{S'}$$

Coordinate systems (Poole 6.3)

Thm Let $S \subset \mathbb{R}^n$ be a subspace and $\mathcal{B} = \{\vec b_1, \dots, \vec b_p\}$ a basis for S. Then for every vector $\vec v \in S$, there is exactly one way to write $\vec v$ as a linear combination of the basis vectors in $\mathcal{B}$:

$$\boxed{\vec v = c_1 \vec b_1 + \dots + c_p \vec b_p.} \quad (*)$$

┌ Let
$$\vec v = c_1 \vec b_1 + \dots + c_p \vec b_p$$
$$\vec v = d_1 \vec b_1 + \dots + d_p \vec b_p$$
two ways to write $\vec v$ as a lin. comb., with different coeff's.

Subtracting, we get $\vec 0 = (c_1 - d_1)\vec b_1 + \dots + (c_p - d_p)\vec b_p$.

but $\vec b$'s are lin. indep. ⇒ all coeffs must be zero ⇒ $c_1 = d_1, \dots, c_p = d_p$ ┘

def Coeffs $c_1, \dots, c_p$ in $(*)$ are called the coordinates of $\vec v$ w.r.t. $\mathcal{B}$.

column vector $[\vec v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ is called the coordinate vector of $\vec v$ w.r.t. $\mathcal{B}$.

Ex: $\mathcal{E} = \{\vec e_1, \vec e_2, \vec e_3\}$ stand. basis for $\mathbb{R}^3$. Find the coord. vector of $\vec v = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ w.r.t. $\mathcal{E}$.

Sol:
$$\vec v = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = 3 \underset{\vec e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + (-1) \underset{\vec e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + 5 \underset{\vec e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \Rightarrow [\vec v]_{\mathcal{E}} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

· Generally: for any $\vec v \in \mathbb{R}^n$, $[\vec v]_{\mathcal{E}} = \vec v$

↖ stand. basis in $\mathbb{R}^n$.

Ex: $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ 10 \\ 11 \end{bmatrix}$

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ - basis for $S = \text{span}(\vec{v}_1, \vec{v}_2)$

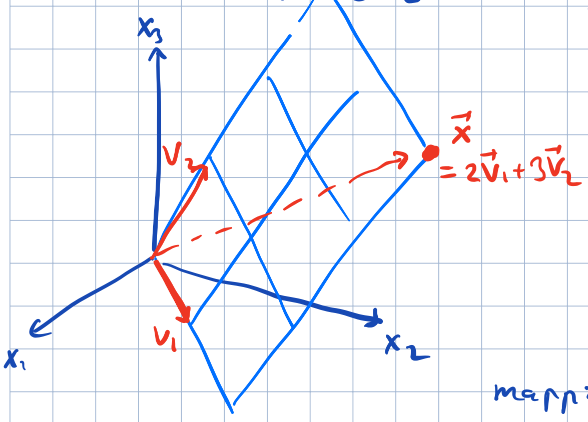
Q: a) is $\vec{x}$ in S?

b) if yes, find $[\vec{x}]_{\mathcal{B}}$

Sol: $\vec{x} \in S$ iff eq. $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$ $(**)$ is consistent.

Aug. Mat. $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 5 & 0 & 10 \\ 1 & 3 & 11 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ $(**)$ consistent, $c_1 = 2, c_2 = 3$ - solution

$\Rightarrow \vec{x} = 2\vec{v}_1 + 3\vec{v}_2 \quad \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

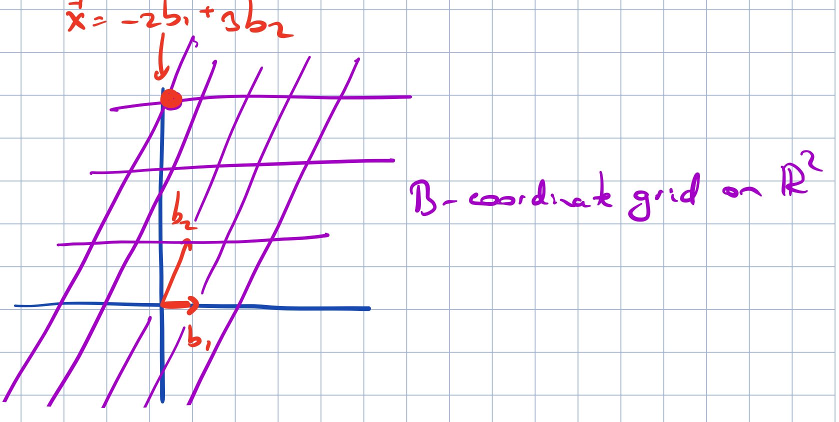


Basis $\mathcal{B}$ determines a coordinate system on S: points of S are in $\mathbb{R}^3$ but are determined by $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^2$.

mapping $S \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$ is called the "coordinate mapping". it is 1-1 and onto, i.e. an "isomorphism"

Ex: $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$ basis for $\mathbb{R}^2$. Find $\vec{x} \in \mathbb{R}^2$ with $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Sol: $\vec{x} = -2\vec{b}_1 + 3\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$



Coordinates in $\mathbb{R}^n$

Ex: basis $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$ for $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ Q: find $[\vec{x}]_{\mathcal{B}}$

Sol: $c_1\vec b_1 + c_2\vec b_2 = \vec x \iff \underbrace{\begin{array}{c}\begin{array}{cc}\vec b_1 & \vec b_2\end{array}\\ \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\end{array}}_{P_{\mathcal B}}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \overset{\vec x}{\begin{bmatrix}4\\ 5\end{bmatrix}}$

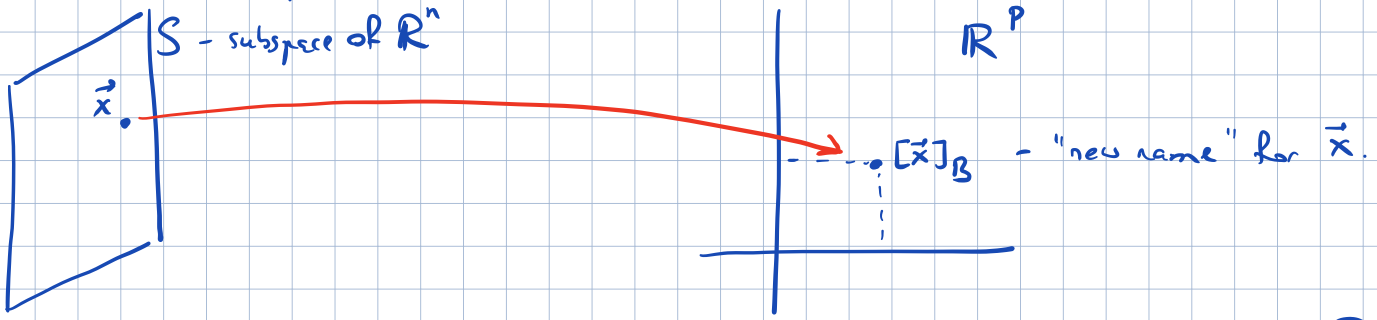
$P_{\mathcal B}$ - matrix changing $\mathcal B$-coordinates of $\vec x$ to stand. coordinates.

solution: $[\vec x]_{\mathcal B} = \begin{bmatrix}c_1\\ c_2\end{bmatrix} = P_{\mathcal B}^{-1}\,\vec x = \begin{bmatrix}3\\ 2\end{bmatrix}$

For any basis $\mathcal B$ for $\mathbb R^n$, let $P_{\mathcal B} = [\vec b_1 \cdots \vec b_n]$.

Then $\vec x = c_1\vec b_1 + \dots + c_n\vec b_n \iff \vec x = \underbrace{P_{\mathcal B}}_{\substack{\text{change-of-coordinates}\\ \text{matrix from } \mathcal B \text{ to } \mathcal E}}[\vec x]_{\mathcal B} \Rightarrow [\vec x]_{\mathcal B} = \underbrace{P_{\mathcal B}^{-1}}_{\substack{\text{matrix of}\\ \text{"coordinate mapping"}\\ \vec x \to [\vec x]_{\mathcal B}}}\vec x.$

Coordinate mapping



Let $\mathcal B$ be a basis for S. Then the coordinate mapping $[\ ]_{\mathcal B}: S \to \mathbb R^p$, $\vec x \mapsto [\vec x]_{\mathcal B}$ is a 1-1 lin. transf. from S onto $\mathbb R^p$.

- A linear mapping $T: \underset{\underset{\mathbb R^n}{\cap}}{S} \to \underset{\underset{\mathbb R^m}{\cap}}{S'}$ - subspaces which is 1-1 and onto is called an isomorphism.

Every vector calculation in S is repeated in S' and vice versa,

So S and S' are "the same".

Thus, a subspace S with a basis of p vectors is indistinguishable from $\mathbb R^p$.