Kernel and range (Poole 6.5)

For a lin. transf. $T: V \to W$,

$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}$ $\underset{\text{subspace}}{\subset} W$

$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$ $\underset{\text{subspace}}{\subset} V$

LAST TIME

<u>def</u> For $T: V \to W$ lin. transf.,

$\text{rank}(T) := \dim \text{range}(T)$

$\text{nullity}(T) := \dim \text{null}(T)$

<u>Rank-nullity Thm</u>: For $T: V \to W$, $\text{rank}(T) + \text{nullity}(T) = \dim V$

<u>Ex</u>: for A $m \times n$ matrix, $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{v} \mapsto A\vec{v}$

$\text{range}(T) = \text{col}(A)$

$\ker(T) = \text{null}(A)$

<u>Ex</u>: $D: P_3 \to P_2$, $p(x) \mapsto p'(x)$ find ker, range, rank, nullity

<u>Sol</u>: $\ker(D) = \{\underset{a+bx+cx^2+dx^3}{\overset{\shortparallel}{p(x)}} \in P_3 \mid D(p) = 0\} = \{a\}$ ← constant polynomials $\Rightarrow$ nullity $= \dim \ker = 1$

$\text{range}(D) = \{\underset{a+bx+cx^2}{\overset{\shortparallel}{p(x)}} \in P_2 \mid p(x) = q'(x) \text{ for some } q(x) \in P_3\} = \text{entire } P_2$ (i.e. D is "onto")

$= \frac{d}{dx}\underbrace{\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}_{q(x)}$

$\Rightarrow$ rank $= \dim$ range $= 3$.

<u>def</u> $T: V \to W$ is "one-to-one" if T maps distinct vectors in V to distinct vectors in W

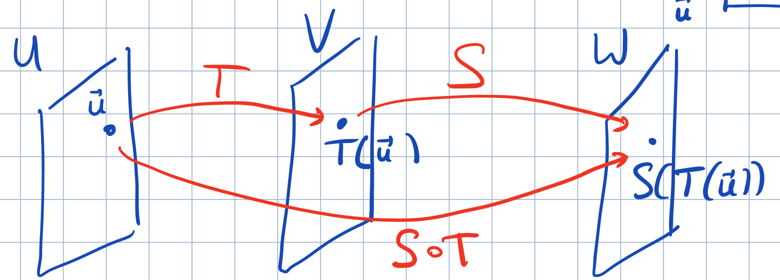
T is "onto" if $\text{range}(T) = W$.

<u>Thm</u>
- $T: V \to W$ is one-to-one iff $\ker(T) = \{\vec{0}\}$
- if $V = W$, $T: V \to W$ is 1-1 iff it is onto.
- if $T: V \to W$ is 1-1, image of a lin. indep. set in V is a lin. indep. set in W.

<u>def</u> A lin. transf. $T: V \to W$ that is <u>1-1</u> and <u>onto</u> is called an <u>isomorphism</u>.

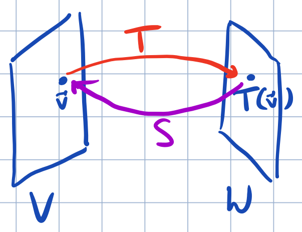
- If $T: U \to V$, $S: V \to W$ lin. transf,

can form the composition $S \circ T : U \to W$,
$\vec{u} \longmapsto S(T(\vec{u}))$



- $T: V \to W$ is <u>invertible</u> if there exists $S: W \to V$ s.t. $\begin{cases} S \circ T = I_V \\ T \circ S = I_W \end{cases}$

Then $S =: T^{-1}$ is the <u>inverse</u> transformation of T.



- T is invertible iff it is an isomorphism.

Ex: $T: P_n \to \mathbb{R}^{n+1}$ — isomorphism

$a_0 + a_1 x + \dots + a_n x^n \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$

Ex: $T: M_{22} \to P_3$ — isomorphism

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto a + bx + cx^2 + dx^3$

- Two vector spaces V, W are <u>isomorphic</u> (an isomorphism $T: V \to W$ exists) iff $\boxed{\dim V = \dim W}$

Coordinates (in a vector space) (Poole 6.2)

Let V be a v.space with a basis $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$.

There is a unique way to write any $\vec{v}\in V$ as a lin. comb. of $\vec{v}_1,\dots,\vec{v}_n$:

$\vec{v}=c_1\vec{v}_1+\dots+c_n\vec{v}_n$. Then, $c_1,\dots,c_n$ are called the coordinates of $\vec{v}$ w.r.t. $\mathcal{B}$,

and the column vector $[\vec{v}]_{\mathcal{B}}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the coordinate vector of $\vec{v}$ w.r.t. $\mathcal{B}$.

Rem If $\dim V=n$, then $[\vec{v}]_{\mathcal{B}}\in\mathbb{R}^n$.

Ex: $p(x)=3-2x+7x^2\in\mathcal{P}_2$, $\mathcal{B}=\{1,x,x^2\}$ - stand. basis for $\mathcal{P}_2$

then: $[p(x)]_{\mathcal{B}}=\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}\in\mathbb{R}^3$

Note: if we change the order of basis vectors to $\mathcal{B}'=\{x^2,x,1\}$,

the coord. vector will change to $[p(x)]_{\mathcal{B}'}=\begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$

Ex: $A=\begin{bmatrix} 1 & 5 \\ -7 & 2 \end{bmatrix}\in M_{22}$, $\mathcal{B}=\{E_{11},E_{12},E_{21},E_{22}\}$ find $[A]_{\mathcal{B}}$

Sol: $A=1\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}}+5\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}}+(-7)\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}}+2\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}\to[A]_{\mathcal{B}}=\begin{bmatrix} 1 \\ 5 \\ -7 \\ 2 \end{bmatrix}$

Ex: $p(x)=1+2x-x^2$, $\mathcal{C}=\{1+x,\,x+x^2,\,1+x^2\}$ - basis for $\mathcal{P}_2$. Find $[p(x)]_{\mathcal{C}}$

Sol: $c_1(1+x)+c_2(x+x^2)+c_3(1+x^2)=1+2x-x^2$

$$\to\begin{cases} c_1 \qquad +c_3=1 \\ c_1+c_2 \qquad =2 \\ \qquad c_2+c_3=-1 \end{cases}\to\begin{cases} c_1=2 \\ c_2=0 \\ c_3=-1 \end{cases}\to[p(x)]_{\mathcal{C}}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Thm: Let $\mathcal{B}$ be a basis for V. Then:

a) $[\vec{u}+\vec{v}]_{\mathcal{B}}=[\vec{u}]_{\mathcal{B}}+[\vec{v}]_{\mathcal{B}}$

b) $[c\vec{u}]_{\mathcal{B}}=c[\vec{u}]_{\mathcal{B}}$

Thm Let $\mathcal{B}$ be a basis for V. Then a set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in V is lin. indep. iff the set of coord vectors $\{[\vec{u}_1]_{\mathcal{B}}, \dots, [\vec{u}_k]_{\mathcal{B}}\}$ is lin. indep. in $\mathbb{R}^n$.

- Given V - v. sp., $\mathcal{B}$ - basis, one has the coordinate mapping

$$T: \quad V \longrightarrow \mathbb{R}^{n} \quad \leftarrow \dim V$$
$$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$$

- it is an isomorphism.

Change of basis (Poole 6.3)

Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$, $\mathcal{C} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be two bases for a v.sp. V and let

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\vec{u}_1]_{\mathcal{C}} [\vec{u}_2]_{\mathcal{C}} \cdots [\vec{u}_n]_{\mathcal{C}}]$$

the $n\times n$ "change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$."

Then: (a) $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$ for any $\vec{x} \in V$

(b) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the unique matrix P with property $[\vec{x}]_{\mathcal{C}} = P[\vec{x}]_{\mathcal{B}}$ for any $\vec{x} \in V$

(c) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible and $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

Ex: $V = \mathcal{P}_2$, $\mathcal{B} = \{\overset{p_1}{1}, \overset{p_2}{x}, \overset{p_3}{x^2}\}$ $\quad \mathcal{C} = \{\overset{q_1}{1+x}, \overset{q_2}{x+x^2}, \overset{q_3}{1+x^2}\}$

(i) find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ $\qquad$ (ii) find $[1+2x-x^2]_{\mathcal{C}}$

Sol: (i) $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is easy: $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [[q_1]_{\mathcal{B}} [q_2]_{\mathcal{B}} [q_3]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$\Rightarrow \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \underset{\text{Gauss-Jordan}}{=} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

(ii) $[1+2x-x^2]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$