## Change of basis (Poole 6.3)

Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ , $\mathcal{C} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be two bases for a v.sp. $V$ and let

$\underset{C \leftarrow B}{P} = [[\vec{u}_1]_C [\vec{u}_2]_C \cdots [\vec{u}_n]_C]$ the $n \times n$ "change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$."

Then: (a) $[\vec{x}]_C = \underset{C \leftarrow B}{P} [\vec{x}]_{\mathcal{B}}$ for any $\vec{x} \in V$

(b) $\underset{C \leftarrow B}{P}$ is the unique matrix $P$ with property $[\vec{x}]_C = P[\vec{x}]_{\mathcal{B}}$ for any $\vec{x} \in V$

(c) $\underset{C \leftarrow B}{P}$ is invertible and $\left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$

Ex: $V = \mathcal{P}_2$ , $\mathcal{B} = \{\overset{p_1}{1}, \overset{p_2}{x}, \overset{p_3}{x^2}\}$ $C = \{\overset{q_1}{1+x}, \overset{q_2}{x+x^2}, \overset{q_3}{1+x^2}\}$

(i) find $\underset{C \leftarrow B}{P}$ , $\underset{B \leftarrow C}{P}$ (ii) find $[1+2x-x^2]_C$

Sol: (i) $\underset{B \leftarrow C}{P}$ is easy: $\underset{B \leftarrow C}{P} = [[q_1]_{\mathcal{B}} [q_2]_{\mathcal{B}} [q_3]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$\Rightarrow \underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \underset{\text{Gauss-Jordan}}{\overset{=}{\uparrow}} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

(ii) $$[1+2x-x^2]_C = \underset{C \leftarrow B}{P} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

# Gauss-Jordan method for finding the change-of-basis matrix

**Thm** Let $\mathcal{B}=\{\vec{u}_1,\dots,\vec{u}_n\}$ and $\mathcal{C}=\{\vec{v}_1,\dots,\vec{v}_n\}$ be two bases for a v.sp. $V$

Let $\underset{= {}_{\mathcal{E}\leftarrow\mathcal{B}}P}{\underline{B}} = \left[\,[\vec{u}_1]_{\mathcal{E}} \cdots [\vec{u}_n]_{\mathcal{E}}\,\right]$ and $\underset{= {}_{\mathcal{E}\leftarrow\mathcal{C}}P}{\underline{C}} = \left[\,[\vec{v}_1]_{\mathcal{E}} \cdots [\vec{v}_n]_{\mathcal{E}}\,\right]$ with $\mathcal{E}$ any basis for $V$.

Then RREF of the $n\times 2n$ matrix $[\,\underline{C} \mid \underline{B}\,]$ is $[\,I \mid {}_{\mathcal{C}\leftarrow\mathcal{B}}P\,]$

**Rem** this is particularly useful for $\mathcal{E}$ = standard basis.

**Ex:** In $M_{22}$, $\mathcal{B}=\{E_{11},E_{21},E_{12},E_{22}\}$,

$$\mathcal{C}=\left\{\underset{A}{\begin{bmatrix}1&0\\0&0\end{bmatrix}},\ \underset{B}{\begin{bmatrix}1&1\\0&0\end{bmatrix}},\ \underset{C}{\begin{bmatrix}1&1\\1&0\end{bmatrix}},\ \underset{D}{\begin{bmatrix}1&1\\1&1\end{bmatrix}}\right\}$$

1) Find ${}_{\mathcal{C}\leftarrow\mathcal{B}}P$.  2) find $\left[\begin{bmatrix}1&2\\3&4\end{bmatrix}\right]_{\mathcal{C}}$

**Sol:** 1) Set $\mathcal{E}=\{E_{11},E_{12},E_{21},E_{22}\}$ standard basis

$$\underline{C}=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix} \qquad \underline{B}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

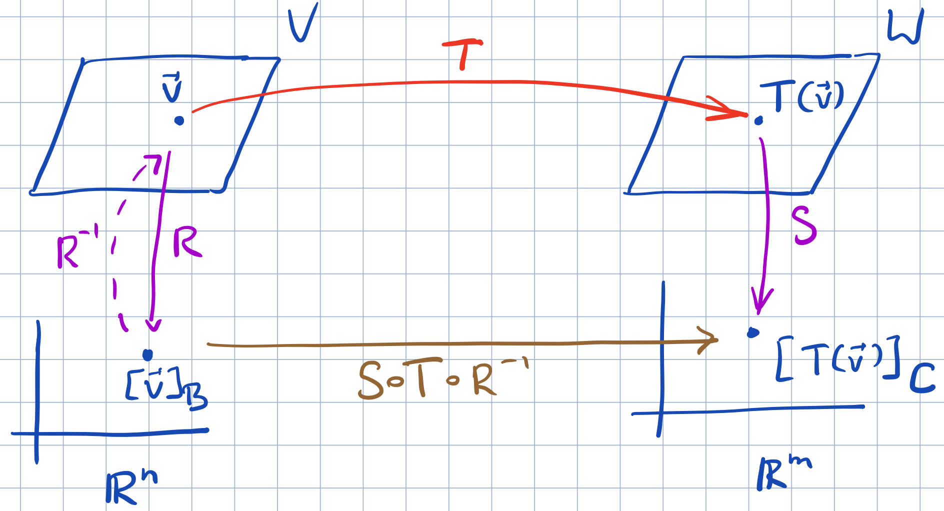
$$[\,\underline{C}\mid\underline{B}\,]=\left[\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&1&1&1&0&0&1&0\\0&0&1&1&0&1&0&0\\0&0&0&1&0&0&0&1\end{array}\right]\xrightarrow[R_1-R_4]{R_3-R_4,\ R_2-R_4}\left[\begin{array}{cccc|cccc}1&1&1&0&1&0&0&-1\\0&1&1&0&0&0&1&-1\\0&0&1&0&0&1&0&-1\\0&0&0&1&0&0&0&1\end{array}\right]$$

$$\xrightarrow[R_1-R_3]{R_2-R_3}\left[\begin{array}{cccc|cccc}1&1&0&0&1&-1&0&0\\0&1&0&0&0&-1&1&0\\0&0&1&0&0&1&0&-1\\0&0&0&1&0&0&0&1\end{array}\right]\xrightarrow{R_1-R_2}\left[\begin{array}{c|c}\underbrace{\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}}_{I} & \underbrace{\boxed{\begin{matrix}1&0&-1&0\\0&-1&1&0\\0&1&0&-1\\0&0&0&1\end{matrix}}}_{{}_{\mathcal{C}\leftarrow\mathcal{B}}P}\end{array}\right]$$

2) $$\left[\begin{bmatrix}1&2\\3&4\end{bmatrix}\right]_{\mathcal{C}} = {}_{\mathcal{C}\leftarrow\mathcal{B}}P\ \underbrace{\left[\begin{bmatrix}1&2\\3&4\end{bmatrix}\right]_{\mathcal{B}}}_{\begin{bmatrix}1\\3\\2\\4\end{bmatrix}} = \begin{bmatrix}-1\\-1\\-1\\4\end{bmatrix}$$

Rem: Alternative way $\underset{C\leftarrow B}{P} = \underset{C\leftarrow \mathcal{E}}{P}\,\underset{\mathcal{E}\leftarrow B}{P} = \left(\underset{\mathcal{E}\leftarrow C}{P}\right)^{-1} \underset{\mathcal{E}\leftarrow B}{P} = \underline{C}^{-1}\underline{B}$

## Matrix of a linear transformation (Poole 6.6)




matrix of $\underbrace{S\circ T\circ R^{-1}}_{T'} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ ?

$A = [\,T'(\vec{e}_1)\ T'(\vec{e}_2)\ \cdots\ T'(\vec{e}_n)\,] = \boxed{[\,[T(\vec{v}_1)]_C \cdots [T(\vec{v}_n)]_C\,]}$

stand. unit vectors in $\mathbb{R}^n$ — basis vectors of $\mathcal{B}$

**Thm** Let $V, W$ be two fin. dim. vector spaces with bases $\mathcal{B}$ and $C$, respectively. Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$. If $T: V \to W$ is a lin. transf., then the $m\times n$ matrix

$A = [\,[T(\vec{v}_1)]_C \cdots [T(\vec{v}_n)]_C\,]$ satisfies

$\underline{A[\vec{v}]_B = [T(\vec{v})]_C}$ for every $\vec{v}\in V$.

Notation: $A = \underset{C\leftarrow B}{[T]}$, thus $[T(\vec{v})]_C = \underset{C\leftarrow B}{[T]}\,[\vec{v}]_B$

- In the case $V=W$ and $\mathcal{B}=C$, special notation $[T]_B := \underset{B\leftarrow B}{[T]}$

**Ex:** $D : P_3 \to P_2$, $p(x) \mapsto p'(x)$  $\mathcal{B} = \{1, x, x^2, x^3\}$ basis for $P_3$

$C = \{1, x, x^2\}$ basis for $P_2$

a) Find $\underset{C \leftarrow \mathcal{B}}{[D]}$

**Sol:** $\underset{C \leftarrow \mathcal{B}}{[D]} = \left[ [\underbrace{1'}_{0}]_C \;\; [\underbrace{x'}_{1}]_C \;\; [\underbrace{(x^2)'}_{2x}]_C \;\; [\underbrace{(x^3)'}_{3x^2}]_C \right]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A$$

b) use $\mathcal{B}' = \{x^3, x^2, x, 1\}$ for $P_3$ instead

$$\underset{C \leftarrow \mathcal{B}'}{[D]} = \left[ [(x^3)']_C \; [(x^2)']_C \; [(x)']_C \, , \; [1']_C \right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

c) $[D(\underbrace{5 - x + 2x^3}_{p(x)})]_C$ $\xrightarrow{\text{can obtain directly}}$ $D(p(x)) = -1 + 6x^2$

$\Rightarrow [D(p(x))]_C = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$

$\downarrow$ can obtain as $A[p(x)]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$ ✓

**Ex:** $T : P_2 \to P_2$, $p(x) \mapsto p(2x-1)$  $\mathcal{B} = \{1, x, x^2\}$

$$[T]_{\mathcal{B}} = \left[ [\underbrace{T(1)}_{1}]_{\mathcal{B}} \;\; [\underbrace{T(x)}_{2x-1}]_{\mathcal{B}} \;\; [\underbrace{T(x^2)}_{4x^2-2x+1}]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

**Rem** Consider $I : V \to V$. Then $\underset{C \leftarrow \mathcal{B}}{[I]} = \underset{C \leftarrow \mathcal{B}}{P}$

$\mathcal{B}$ $C$ — two bases for $V$

changes $[\vec{v}]_{\mathcal{B}}$ to $[\vec{v}]_C$