Properties of determinants

- $\det(AB) = (\det A)(\det B)$

  Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ $\qquad AB = \begin{bmatrix} 2 & 5 \\ 4 & 13 \end{bmatrix}$

  $\det A = 3 \qquad \det B = 2 \qquad \det(AB) = 26 - 20 = 6$

  $= \det A \cdot \det B$ ✓

Corollary: $\det A^{-1} = \dfrac{1}{\det A}$ for A invertible

- A is invertible iff $\det A \neq 0$

  ($\det A = 0 \iff$ columns of A form a lin. dep. set
  $\iff$ rows of A form a lin. dep. set)

WARNING: $\det(A+B) \neq \det A + \det B$ generally

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad 1 = \det \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A+B} \neq \underbrace{\det A}_{0} + \underbrace{\det B}_{0}$

- $\det A^T = \det A$
- $\det(cA) = c^n \det A$ (not $c \det A$!)
- det is linear in i-th column (row):

$T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\vec{x} \longmapsto \det[\underbrace{\vec{a}_1 \cdots \vec{a}_{i-1}}\ \vec{x}\ \underbrace{\vec{a}_{i+1} \cdots \vec{a}_n}]$

fixed vectors in $\mathbb{R}^n$

is a linear mapping:

$T(c\vec{x}) = c\,T(\vec{x})$

$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

Cramer's rule (Poole 4.2)

For $A = [\vec{a}_1 \cdots \vec{a}_n]$ an $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$,

$\uparrow \quad \uparrow$ columns

denote $A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n]$ (column i in A is replaced by $\vec{b}$)

$\uparrow$ column i

Thm (Cramer's rule)

Let A be an invertible $n \times n$ matrix and let $\vec{b} \in \mathbb{R}^n$. Then the unique solution $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of the system $A\vec{x} = \vec{b}$ is given by

$$\boxed{x_i = \frac{\det A_i(\vec{b})}{\det A}}, \quad i = 1, \ldots, n$$

Ex: $\begin{aligned} 4x_1 + 5x_2 &= 2 \\ 2x_1 + 3x_2 &= 6 \end{aligned}$ solve using Cramer's rule

Sol: $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $A_1(\vec{b}) = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$ $A_2(\vec{b}) = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$

$\det A = 2$ $\qquad \det = -24$ $\qquad \det = 20$

$$x_1 = \frac{\det A_1(\vec{b})}{\det A} = \frac{-24}{2} = -12, \qquad x_2 = \frac{\det A_2(\vec{b})}{\det A} = \frac{20}{2} = 10$$

Ex: For which values of parameter s, the system $\begin{aligned} 3s x_1 - 2x_2 &= 1 \\ -6x_1 + s x_2 &= 2 \end{aligned}$

(a) has a unique solution

(b) write the solution using Cramer's rule.

Sol: $A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$ $\det A = 3s^2 - 12 = 3(s-2)(s+2)$

(a) $\det A \neq 0$ iff $s \neq \pm 2$

(b) $A_1(\vec{b}) = \begin{bmatrix} 1 & -2 \\ 2 & s \end{bmatrix}$, det $= s+4$ $\qquad$ $A_2(\vec{b}) = \begin{bmatrix} 3s & 1 \\ -6 & 2 \end{bmatrix}$, det $= 6s+6 = 6(s+1)$

So: $x_1 = \dfrac{s+4}{3(s-2)(s+2)}$, $\quad x_2 = \dfrac{6(s+1)}{3(s-2)(s+2)} = \dfrac{2(s+1)}{(s-2)(s+2)}$

Formula for A^{-1}

- for A an $n\times n$ matrix, the matrix $\underset{\text{cofactors}}{[C_{ji}]} = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$ is called the "adjoint" (or "adjugate") of A and denoted adj A

Thm Let A be an invertible $n\times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\,A.$$

Equivalently, $\boxed{(A^{-1})_{ij} = \dfrac{C_{ji}}{\det A}}$

Ex: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & -6 \end{bmatrix}$ find $(A^{-1})_{12}$

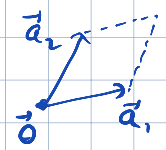
Sol: $(A^{-1})_{12} = \dfrac{C_{21}}{\det A}$

$\det A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & -6 \end{vmatrix} \underset{R_3-2R_2}{=} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -6 \end{vmatrix} = 7$ $\qquad$ So, $(A^{-1})_{12} = \dfrac{7}{1} = 7$

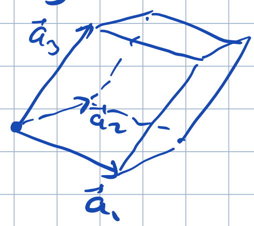
Determinants as area or volume

Thm (a) if $A = [\vec{a}_1\ \vec{a}_2]$ is a 2×2 matrix, the area of the parallelogram determined by $\vec{a}_1, \vec{a}_2$ is $|\det A|$

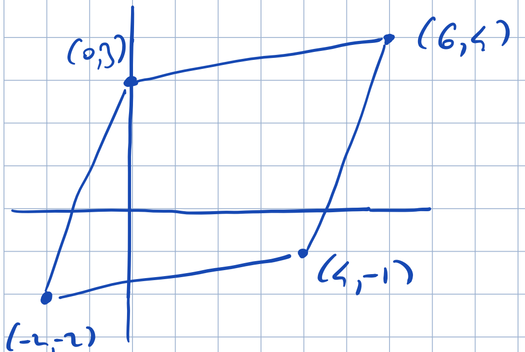


(b) if $A = [\vec{a}_1 \vec{a}_2 \vec{a}_3]$ is a 3×3 matrix,

the volume of the parallelipiped determined by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is $|\det A|$

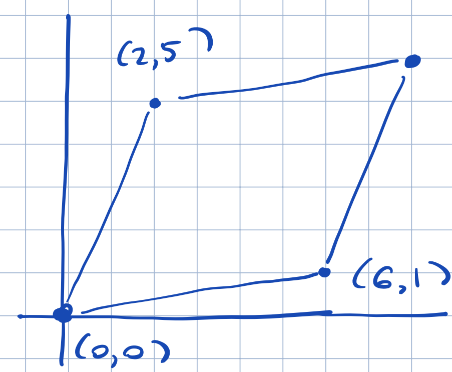


Ex: Find the area of the parallelogram with vertices at $(-2,-2)$, $(0,3)$, $(4,-1)$, $(6,4)$



Sol: translate the parallelogram by $(2,2)$ to have $\vec{0}$ as a vertex

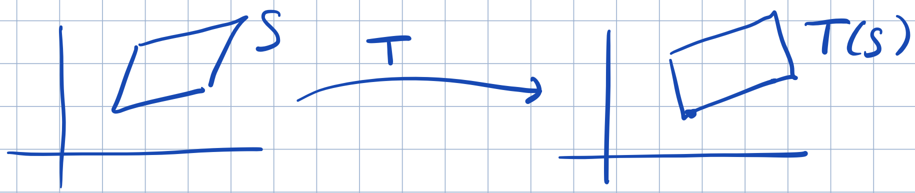
new parall:



$$\text{Area} = \Big|\underbrace{\det\begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}}_{-28}\Big| = 28$$

Thm* (a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. transf. determined by a 2×2 matrix A.
If S is a parallelogram in $\mathbb{R}^2$, then $\text{Area}(T(S)) = |\det A| \cdot \text{Area}(S)$

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a lin. transf. determined by a 3×3 matrix A.
If S is a parallelipiped in $\mathbb{R}^3$, then $\text{Volume}(T(S)) = |\det A| \cdot \text{Volume}(S)$



- In fact, Thm* generalizes to finite-area regions S of $\mathbb{R}^2$ / finite-volume regions S of $\mathbb{R}^3$

- <u>Corollary:</u> if $\det A = \pm 1$, then T preserves areas/volumes