LAST TIME

Thm ("Diagonalization thm")

An $n \times n$ matrix A is diagonalizable iff A has n lin. indep. eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, Then,

$\lambda_1 \ldots \lambda_n$ - corresp. e.v.

$$A = PDP^{-1} \quad \text{with } D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad P = [\vec{v}_1 \cdots \vec{v}_n]$$

Case of non-distinct eigenvalues

Thm Let A be $n \times n$ mat. whose distinct e.v. are $\lambda_1, \dots, \lambda_p$

$m_1, \dots, m_p$ - alg. multiplicities

(a) for each $k = 1, \dots, p$, the dimension $\underset{\uparrow}{d_k}$ of λ_k-eigenspace is $\leq m_k$

"geometric multiplicity" of the e.v. λ_k

(b) A is diagonalizable iff $d_k = m_k$ for all k. $\left(\Longleftrightarrow \sum_{k=1}^{p} d_k = n \right)$

(c) If A is diagonalizable and B_k - basis for E_{λ_k}, then

$B_1 \cup B_2 \cup \dots \cup B_p$ - basis of eigenvectors for $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$ Q: diagonalize if possible

Sol: $\lambda = 5, \ -3$ eigenvalues

$2 \quad 2$ alg. multiplicities

$$\left.\begin{array}{l} \text{basis for } E_5: \quad \vec{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \qquad \text{geom. mult.} = 2 \\ \text{basis for } E_{-3}: \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{geom. mult} = 2 \end{array}\right\} \underset{(b)}{\Longrightarrow} A \text{ diagonalizable}$$

by (c), $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

So: $A = PDP^{-1}$, $P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

Linear transformations and diagonalization

Thm Let $A = PDP^{-1}$ with D a diagonal matrix.
Let $\mathcal{B}$ be the basis for $\mathbb{R}^n$ formed from the columns of P.
Then the matrix of $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$ w.r.t. $\mathcal{B}$ is D.

⌜ $$[T]_{\mathcal{B}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}} = \underbrace{P_{\mathcal{B}\leftarrow\mathcal{E}}}_{P^{-1}} \underbrace{[T]_{\mathcal{E}\leftarrow\mathcal{E}}}_{A} \underbrace{P_{\mathcal{E}\leftarrow\mathcal{B}}}_{P} \underset{\substack{\uparrow \\ \text{using } A = PDP^{-1}}}{=} D$$ ⌟

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto A\vec{x}$, $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ find a basis $\mathcal{B}$ for $\mathbb{R}^2$ s.t. $[T]_{\mathcal{B}}$ is diagonal.

Sol: $A = PDP^{-1}$ with $P = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \\ \vec{b}_1 \quad \vec{b}_2 \end{array}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Thus, for $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $[T]_{\mathcal{B}} = D$.

I.e. mappings $\vec{x} \longmapsto A\vec{x}$ and $\vec{u} \longmapsto D\vec{u}$ describe the same lin. transf. w.r.t. different bases.

Note: Thm above has a generalization:

if $A \sim \underset{\substack{\uparrow \\ \text{not necessarily} \\ \text{diagonal}}}{C}$, i.e., $A = PCP^{-1}$, and $T: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$, then

$$\boxed{[T]_{\mathcal{B}} = C}$$

$\mathcal{B}$: basis formed out of columns of P.

$$\begin{tikzcd} \vec{x} \arrow[r, "\text{mult. by}"', "A"'] \arrow[d, "\text{mult. by } P^{-1}"] & A\vec{x} \\ {[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{mult. by}", "C"'] & {[A\vec{x}]_{\mathcal{B}}} \arrow[u, "\text{mult. by } P"'] \end{tikzcd}$$

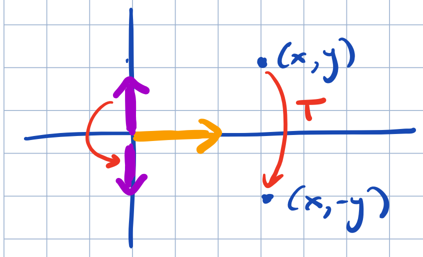
Conversely, the matrix of T w.r.t. any basis $\mathcal{B}$ is similar to A.

def. A lin. transf $T: \overset{\text{v. space}}{V \to V}$ is "diagonalizable" if $[T]_{\mathcal{B}}$ is a diagonal matrix for some basis $\mathcal{B}$ for V.

- Sometimes we can find eigenvectors and eigenvalues geometrically.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ - matrix of the lin. transf. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ - reflection in x-axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix}$$



vectors which T maps parallel to themselves, $T\vec{v} = \lambda \vec{v}$ are

- $\vec{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} \xmapsto{T} \vec{v}$, $\lambda = 1$

vectors parallel to x-axis

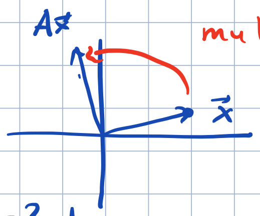
- $\vec{v} = \begin{bmatrix} 0 \\ y \end{bmatrix} \xmapsto{T} -\vec{v}$, $\lambda = -1$

vectors parallel to y-axis

So: $\lambda = 1$, $\lambda = -1$ eigenvalues, $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

Complex eigenvalues

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



mult. by A = rotation by 90°

A has no eigenvectors in $\mathbb{R}^2$!

char. eq.: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \boxed{\lambda^2 + 1 = 0}$ complex roots: $\lambda = i$, $\lambda = -i$

If we allow A to act on $\mathbb{C}^2$:

$$A \underbrace{\begin{bmatrix} 1 \\ -i \end{bmatrix}}_{\text{eigenvector for } \lambda = i} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \boxed{i} \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad A \underbrace{\begin{bmatrix} 1 \\ i \end{bmatrix}}_{\text{eigenvector for } \lambda = -i} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \boxed{-i} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Ex*: $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ Q: find eigenvalues & eigenvectors

Sol: char eq. $0 = \begin{vmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 1.6\lambda + 1$ solutions: $\lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4}}{2}$

$= 0.8 \pm 0.6i$

for $\lambda = 0.8 - 0.6i$,

$A - \lambda I = \begin{bmatrix} -0.3+0.6i & -0.6 \\ 0.75 & 0.3+0.6i \end{bmatrix}$ (1) $(-0.3+0.6i)x_1 - 0.6x_2 = 0$

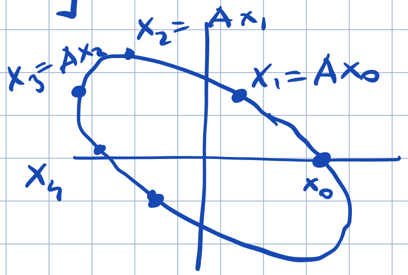
(2) $0.75x_1 + (0.3+0.6i)x_2 = 0$

nontriv. sol. exists => both eqs determine the same relation between x_1 and x_2, (1) <=> (2)

<=> $x_1 = -(0.4+0.8i)\, x_2$ choose $x_2 = 5$ => basis for the eigenspace: $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

similarly, for $\lambda = 0.8+0.6i$, eigenvector $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$

- mapping $\vec{x} \longmapsto A\vec{x}$ is "essentially" a rotation:



- for A a matrix with real entries,

$$A\vec{x} = \lambda\vec{x} \implies A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$$

complex conjugation $(\overline{a+ib} = a - ib)$

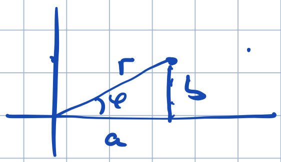
so: complex eigenvalues $\lambda = a+ib$, $b \neq 0$ occur in conjugate pairs.

In Ex*: $\lambda = 0.8 - 0.6i$ $\overline{\lambda} = 0.8 + 0.6i$ - conjugate

$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$ - conjugate

Ex: $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a, b real, nonzero. Eigenvalues: $\lambda = a \pm ib$ and

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling by } r = |\lambda| = \sqrt{a^2+b^2}} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation by } \varphi = \text{argument of } a+ib}.$$



Back to Ex* $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ $\lambda = 0.8 - 0.6i$ $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

Let $P = [\operatorname{Re} \vec{v}_1 \quad \operatorname{Im} \vec{v}_1] = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$

Let $C = P^{-1} A P = \dots = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ - pure rotation by $\varphi = \arctan \frac{0.6}{0.8}$

since $|\lambda| = \sqrt{0.8^2 + 0.6^2} = 1$

Thus: $A = P \underbrace{C}_{\text{rotation}} P^{-1}$

$\vec{x} = P\vec{u}$

change of variable

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, "P^{-1}", "\text{change of var.}"'] & A\vec{x} \\
\vec{u} \arrow[r, "C", "\text{rotation}"'] & C\vec{u} \arrow[u, "P"', "\text{change of var.}"]
\end{tikzcd}$$

Thm Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a - ib$, $b \neq 0$ and $\vec{v}$ the corresp. eigenvector in $\mathbb{C}^2$. Then

$$A = PCP^{-1} \quad \text{with } P = [\operatorname{Re} \vec{v} \quad \operatorname{Im} \vec{v}], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$