Ex: $W = \left\{ \underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\} \subset M_{22}$ is a subspace

$\Rightarrow W = \text{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$

- We also interested in sets that span the entire V.

Ex: polynomials $1, x, x^2$ in P_2 span P_2, since each $p(x) \in P_2$
$= a_0 + a_1 x + a_2 x^2$
is a lin. comb. of $1, x, x^2$.

Likewise, $1, x, \dots, x^n$ in P_n span P_n.

Ex: $M_{22} = \text{Span}\left(\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}} \right)$

since $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} E_{11} + a_{12} E_{12} + a_{21} E_{21} + a_{22} E_{22}$

$M_{mn} = \text{Span}\{E_{ij}\}_{\substack{i=1\dots m \\ j=1\dots n}}$ $\qquad E_{ij}$ = matrix where (i,j)-entry is 1
all other entries are 0.

Ex: $V = \mathbb{R}^2$ with operations $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{\text{new addition}}{\oplus} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 1 \\ x_2 + y_2 \end{bmatrix}$,

$c \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c x_1 + c - 1 \\ c x_2 \end{bmatrix}$

zero-vector: $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

negative: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -x_1 - 1 \\ -x_2 \end{bmatrix}$

– a vector space!

- Does a vector belong to a given span?

Ex: In P_2, is $r(x) = 1-4x+6x^2$ in $\operatorname{span}(\underbrace{1-x+x^2}_{p(x)}, \underbrace{2+x-3x^2}_{q(x)})$?

Sol: We want scalars c, d s.t.

$$c\,p(x) + d\,q(x) = r(x)$$

i.e. $(c+2d) + (-c+d)x + (c-3d)x^2 = 1 - 4x + 6x^2$

i.e.
$$\begin{aligned} c + 2d &= 1 \\ -c + d &= -4 \\ c - 3d &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c &= 3 \\ d &= -1 \end{aligned} \quad \Rightarrow \quad r(x) = 3p(x) - q(x)$$

Ex: Does the set $\{p(x), q(x)\}$ span P_2?

Sol: it spans P_2 iff eq. $c\,p(x) + d\,q(x) = f(x)$ is consistent for any $f(x) = s + tx + ux^2 \in P_2$

$$\begin{aligned} c + 2d &= s \\ -c + d &= t \\ c - 3d &= u \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 2 & s \\ -1 & 1 & t \\ 1 & -3 & u \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{cc|c} 1 & 2 & s \\ 0 & 3 & s+t \\ 0 & 0 & 2s+5t+3u \end{array}\right]}_{\text{REF}}$$

- only consistent when vanishes

$\Rightarrow \{p(x), q(x)\}$ does not span P_2.

< another way: coeff. mat. is 3×2, so cannot have a pivot in every row, so $A\vec{x} = \vec{b}$ cannot be consistent for every $\vec{b}$ >

Ex: In $\mathcal{F} = \{\text{functions on } \mathbb{R}\}$, determine whether $\sin 2x$ is in $W = \operatorname{span}(\sin x, \cos x)$

Sol: assume $\sin 2x = c \sin x + d \cos x$ - then it should be true for all values of x.

$x = 0 \Rightarrow 0 = 0\cdot c + 1\cdot d \Rightarrow d = 0$

$x = \frac{\pi}{2} \Rightarrow 0 = 1\cdot c + 0\cdot d \Rightarrow c = 0$

$\Rightarrow \sin 2x = 0\cdot \sin x + 0\cdot \cos x$ which is wrong $\Rightarrow$ contradiction

$\Rightarrow \sin 2x \notin \operatorname{span}(\sin x, \cos x)$

Linear independence, basis, dimension (Poole 6.2)

def A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space V is <u>linearly dependent</u> if there exist scalars $c_1, \ldots, c_k$ (not all zero) s.t.

$c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$.

Otherwise, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is <u>lin. independent</u>

- A set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ iff some $\vec{v}_j$ can be expressed as a lin. comb. of the others.

Ex: $\{\underset{p}{1}, \underset{q}{x+x^2}, \underset{r}{2+x+x^2}\}$ is lin. dep. in P_2 $(r = 2p+q)$

Ex: $\{\underset{f}{\sin^2 x}, \underset{g}{\cos^2 x}, \underset{h}{\cos 2x}\}$ is lin. dep. in $\mathcal{F}$ $(h = g - f)$

Ex: Is the set $S = \{1+x, x+x^2, 1+x^2\}$ lin. indep. in P_2?

Sol: $c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$

$$\Leftrightarrow \begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases} \Rightarrow \text{the set } S \text{ is } \underline{\text{lin. indep.}}$$

Ex: $\{1, x, \ldots, x^n\}$ is a lin. indep. set in P_n.

Basis

def A subset $\mathcal{B} \subset V$ is a <u>basis</u> for V if

(1) $\mathcal{B}$ spans V

(2) $\mathcal{B}$ is lin. indep.

Ex: $\{\vec{e}_1, \ldots, \vec{e}_n\}$ - stand. basis for $\mathbb{R}^n$

Ex: $\{1, x, \ldots, x^n\}$ - stand. basis for P_n

Ex: $\{E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{m1}, \ldots, E_{mn}\}$ - stand. basis for M_{mn}

Ex: $\mathcal{B} = \{\underset{p_1(x)}{1+x}, \underset{p_2(x)}{x+x^2}, \underset{p_3(x)}{1+x^2}\}$ is a basis for P_2

- we already know that it is lin. indep. Does it span P_2?

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = \underbrace{a + bx + cx^2}_{\text{arbitrary element of } P_2}$$

$$\Leftrightarrow \begin{array}{l} c_1 + c_3 = a \\ c_1 + c_2 = b \\ \phantom{c_1+{}} c_2 + c_3 = c \end{array} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{has rank 3} \Rightarrow \text{invertible}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Rightarrow$ sys. is consistent for any a, b, c

$\Rightarrow \mathcal{B}$ spans P_2. $\Rightarrow \mathcal{B}$ is a basis for P_2.

Ex: $W = \left\{ \underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \middle| \, a, b \in \mathbb{R} \right\} \subset M_{22}$ has a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Dimension

Basis theorem: If a vector space V has a basis of n vectors, then every basis for V has exactly n vectors.

- A v. sp. V is called finite-dimensional if it has a basis consisting of finitely many vectors. Dimension of V ($\dim V$) is the number of vectors in a basis for V.
- $\dim \{\vec{0}\} = 0$ (convention)
- A v. sp. that has no finite basis is called infinite-dimensional.

Ex: $\mathbb{R}^n$ has a basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ $\Rightarrow \dim \mathbb{R}^n = n$

Ex: P_n has a basis $\{1, x, \ldots, x^n\} \Rightarrow \dim P_n = n+1$

Ex: M_{mn} has a basis $\{E_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}} \Rightarrow \dim M_{mn} = mn$

Ex: $\mathcal{P}$ and $\mathcal{F}$ are ∞-dimensional (each contains an infinite lin. dep. set $1, x, x^2, \ldots$)