Thm An $m\times n$ matrix U has o/n columns iff $U^TU = I$

Thm Let U be an $m\times n$ matrix with o/n columns; let $\vec{x},\vec{y}\in\mathbb{R}^n$. Then:

a) $\|U\vec{x}\| = \|\vec{x}\|$ b) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$ c) $U\vec{x}\perp U\vec{y}$ iff $\vec{x}\perp\vec{y}$

i.e. mapping $\underline{\begin{matrix}\mathbb{R}^n \to \mathbb{R}^m \\ \vec{x}\mapsto U\vec{x}\end{matrix}}$ preserves lengths and orthogonality

· If $m=n$, square mat. U with o/n columns is called an orthogonal matrix.
U is orthogonal iff $U^{-1} = U^T$.

Ex: matrix of rotation by angle φ: $U = \begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix}$

$U^{-1} = \cancel{\frac{1}{\cos^2\varphi+\sin^2\varphi}}\begin{bmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{bmatrix} = U^T \Rightarrow U$ is an orthogonal matrix.

Thm If $\{\vec{u}_1,\dots,\vec{u}_p\}$ is an o/n basis for $W\subset\mathbb{R}^n$, then

(1) $\mathrm{proj}_W\vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + \dots + (\vec{v}\cdot\vec{u}_p)\vec{u}_p$.

(2) If $U = [\vec{u}_1\cdots\vec{u}_p]$ then $\boxed{\mathrm{proj}_W\vec{v} = \underbrace{UU^T}_{\text{matrix of projection onto } W.}\vec{v}}$ for all $\vec{v}\in\mathbb{R}^n$

Gram-Schmidt process

Problem: $W \subset \mathbb{R}^n$ find an <u>orthogonal</u> basis for W.

$W = \operatorname{Span}(\underbrace{\vec{x}_1, \dots, \vec{x}_p}_{\text{basis but not orthogonal}})$

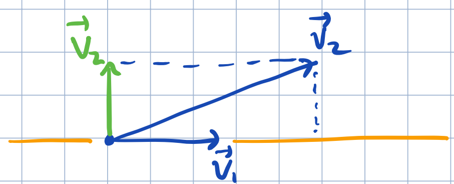
Ex: $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad W = \operatorname{span}(\vec{x}_1, \vec{x}_2) \subset \mathbb{R}^3$

Q: find an orthogonal basis

Sol: set $\vec{v}_1 = \vec{x}_1$; $\vec{v}_2$ in W, lin. indep. from $\vec{v}_1$ and $\perp$ to $\vec{v}_1$

$$\vec{x}_2 = \operatorname{proj}_{\vec{v}_1} \vec{x}_2 + \underbrace{\boxed{\operatorname{perp}_{\vec{v}_1} \vec{x}_2}}_{\perp \vec{v}_1}$$

$\vec{v}_2$



Explicitly: $\operatorname{proj}_{\vec{v}_1} \vec{x}_2 = \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \dfrac{-1}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix}$

$$\operatorname{perp}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{v}_1} \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

So: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix} \right\}$ - orthog. basis for W.

$\Big\}$ rescale

$\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2' = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} \right\}$ - more convenient, also orthogonal

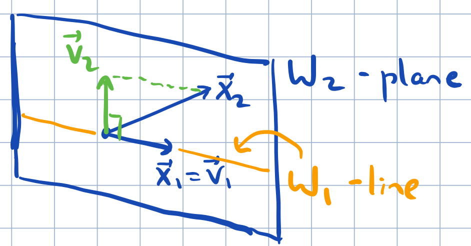
Generally Let $W = \text{span}(\underbrace{\vec{x}_1, \dots, \vec{x}_p}_{\text{basis}})$

want to construct an orthog. basis for W, $\{\vec{v}_1, \dots, \vec{v}_p\}$

Step 1: set $\vec{v}_1 = \vec{x}_1$, $W_1 = \text{span}(\vec{x}_1) = \text{span}(\vec{v}_1)$

Step 2: $W_2 = \text{span}(\vec{x}_1, \vec{x}_2)$ orthog. basis: $\vec{v}_1 = \vec{x}_1$

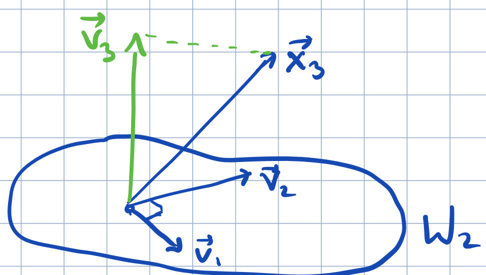
$$\vec{v}_2 = \text{perp}_{W_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$



Step 3: $W_3 = \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ orthog. basis: $\vec{v}_1, \vec{v}_2$ - already constructed

$$\vec{v}_3 = \text{perp}_{W_2} \vec{x}_3 \quad \{\vec{v}_1, \vec{v}_2\} \text{ - orthog. basis}$$

$$= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$



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Step p: $W = W_p = \text{span}(\vec{x}_1, \dots, \vec{x}_p)$

orthog. basis: $\vec{v}_1, \dots, \vec{v}_{p-1}$, $\vec{v}_p = \text{perp}_{W_{p-1}} \vec{x}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$

$\{\vec{v}_1, \dots, \vec{v}_{p-1}\}$ - orthog. basis

Ex*: $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ - basis for $W = \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \subset \mathbb{R}^4$

Q: find an orthogonal basis for W.

Sol: $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{v}_2 = \text{perp}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \xrightarrow[\text{by } 2]{\text{rescale}} \vec{v}_2{}' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \text{perp}_{\text{Span}(\vec{v}_1, \vec{v}_2{}')} \vec{x}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2{}'}{\vec{v}_2{}' \cdot \vec{v}_2{}'} \vec{v}_2{}'$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \cancel{\frac{2}{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \underset{\frac{1}{3}}{\frac{2}{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix} \xrightarrow[\text{rescale}]{} \vec{v}_3{}' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Thus, $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2{}' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3{}' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ - orthogonal basis for W

Q: find an <u>orthonormal</u> basis for W

<u>Sol:</u> normalize $\vec{v}_1, \vec{v}_2{}', \vec{v}_3{}'$ to unit length:

$\left\{ \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ - o/n basis for W.

<u>QR factorization</u>

<u>Thm</u> (QR factorization)

If A is an $m \times n$ matrix with <u>lin. indep. columns</u>, then A can be factored as $A = QR$ where Q is an $m \times n$ matrix whose columns form an o/n basis for $\text{col}(A)$ and R is an $n \times n$ upper-triangular invertible matrix with positive diagonal entries.

Idea: $A = [\vec{x}_1 \cdots \vec{x}_n]$ $\quad W = \text{col}(A) = \text{Span}(\vec{x}_1, \ldots, \vec{x}_n) \subset \mathbb{R}^m$

$\big\downarrow$ Gram-Schmidt + normalization

$\{\vec{u}_1, \ldots \vec{u}_n\}$ - o/n basis for W

$$\vec{x}_k = \underbrace{\text{proj}_{W_{k-1}} x_k}_{=\, r_{1k}\vec{u}_1 + \ldots + r_{k-1\,k}\vec{u}_{k-1}} + \underbrace{\vec{v}_k}_{=\, r_{kk}\vec{u}_k} \quad \leftarrow \text{from Gram-Schmidt}$$

$$= r_{1k}\vec{u}_1 + \ldots + r_{k-1\,k}\vec{u}_{k-1} + r_{kk}\vec{u}_k + 0 \cdot \vec{u}_{k+1} + \ldots + 0 \cdot \vec{u}_n$$

$$\Rightarrow A = \underbrace{[\vec{u}_1 \cdots \vec{u}_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \ddots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots\ 0 & r_{nn} \end{bmatrix}}_{R}$$

$\lrcorner$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find QR factorization

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{matrix}$ - vectors from Ex*

Sol: $Q = [\underbrace{\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3}_{\substack{\text{normalized} \\ \text{G-S basis}}}] = \boxed{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}}$

a short cut to get R: $A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R = R$

So, $R = Q^T A = \dots = \boxed{\begin{bmatrix} 2 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 3/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 4/\sqrt{12} \end{bmatrix}}$