Differential Equations (textbook: Zill 1.1, 1.2)

An equation, containing derivatives of one or more functions (or dependent variables) w.r.t. one or more independent variables, is called a differential equation (DE).

Classification by type:

- If there is one indep. var., it is an ordinary diff. eq. (ODE)

Ex: $\frac{dy}{dx} + 2y = \sin x$, $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$, $\frac{dx}{dt} + \frac{dy}{dt} = 0$ two unknown functions

or: $y' + 2y = \sin x$ ← prime notation

or: $y_x + 2y = \sin x$ ← subscript notation

- If there are ≥ 2 indep. var., the DE is a partial diff. eq. (PDE)

Ex: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

or: $u_{xx} + u_{yy} = 0$

Order of a DE = order of the highest derivative in the DE

Ex: $\underbrace{\frac{d^2y}{dx^2}}_{2^{nd}\text{ order}} + \Big(\underbrace{\frac{dy}{dx}}_{1^{st}\text{ order}}\Big)^3 + y = e^x$ ← 2^{nd} order ODE

- Differential form of a 1^{st} order ODE: $M(x,y)\,dx + N(x,y)\,dy = 0$

Ex: $xy\frac{dy}{dx} + x + y = 0 \underset{\cdot dx}{\iff} \underbrace{(x+y)\,dx + xy\,dy = 0}_{\text{diff. form}}$

- normal form of n-th order ODE:

general n-th order ODE: $F(x, y, y', \dots, y^{(n)}) = 0$ ⟶ solve for $y^{(n)}$

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

f ↑ given function

Ex: $xy\frac{dy}{dx} + x + y = 0 \xrightarrow{\text{solve for } \frac{dy}{dx}} \frac{dy}{dx} = -\frac{x+y}{xy}$ ← normal form

• n-th order ODE is linear if it is of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Ex: $x^2\frac{d^3y}{dx^3} + \sin x\,\frac{dy}{dx} = e^x$ - linear ODE

$yy' + 2y = 3$ - nonlinear ODE

Solutions **def** Given an ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ $(*)$, any function φ on an interval I with $\geq n$ continuous derivatives on I, such that $F(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x)) = 0$ for all $x \in I$, is called a solution of $(*)$ on I.

I is the "interval of definition" of the solution.

Ex: for the ODE $y'' = -y$, $y = \sin x$ is a solution on $I = (-\infty, \infty)$

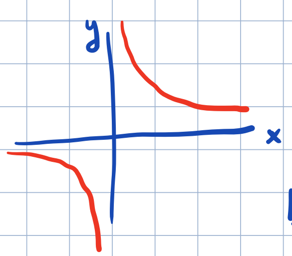
indeed, l.h.s.: $y'' = (\cos x)' = -\sin x$ ✓

r.h.s. $-y = -\sin x$

• "solution curve" = graph of a solution $\varphi(x)$ of ODE.

Ex: (a) for $y = \frac{1}{x}$ (a function), the domain is $x \neq 0$ or: $x \in (-\infty, 0) \cup (0, \infty)$

at $x = 0$ the function is discontinuous.



(b) $y = \frac{1}{x}$ is a solution of $xy' + y = 0$ $(**)$

but as a solution of $(**)$ it is defined on any single interval where it is differentiable and satisfies $(**)$, e.g. on $(-3, -1)$.

Largest possible intervals: $I = (-\infty, 0)$

and $I = (0, \infty)$

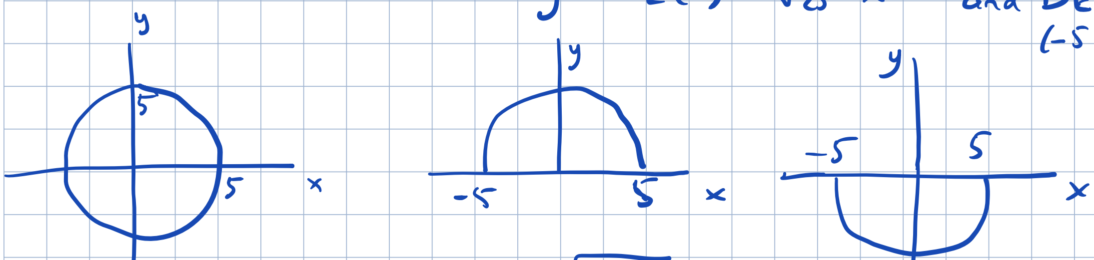
• $y = \varphi(x)$ - explicit solution

$G(x, y) = 0$ - implicit solution on I, if there is at least one function φ satisfying $G(x, \varphi(x)) = 0$ and the DE on I.

Ex: $x^2+y^2=25 \quad (\#)$ is an implicit sol. of $\frac{dy}{dx}=-\frac{x}{y}$ on $(-5,5)$

Check: · $\frac{d}{dx}(\#) \Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$ ✓

· solving $(\#)$ for y: $y=\varphi_1(x)=\sqrt{25-x^2}$, $y=\varphi_2(x)=-\sqrt{25-x^2}$ two functions satisfying $(\#)$ and DE on $(-5,5)$ - two explicit solutions on $(-5,5)$



implicit solution

$\underbrace{y=\sqrt{25-x^2} \qquad y=-\sqrt{25-x^2}}_{\text{explicit solutions}}$

Ex: $x^2+y^2=C$ - a one-parameter family of (implicit) solutions of $\frac{dy}{dx}=-\frac{x}{y}$

C ← arbitrary constant

$$\begin{cases} \frac{dy}{dx}=-\frac{x}{y} & \leftarrow \text{DE} \\ y(0)=1 & \leftarrow \text{side condition} \end{cases}$$

side condition → singles out the solution with $x^2+y^2=\underbrace{1}_{0^2+1^2}$ - particular solution

Ex: $\underset{\text{ODE}}{y''=-y}$, $y=C_1\sin x+C_2\cos x$ - two-parameter family of solutions

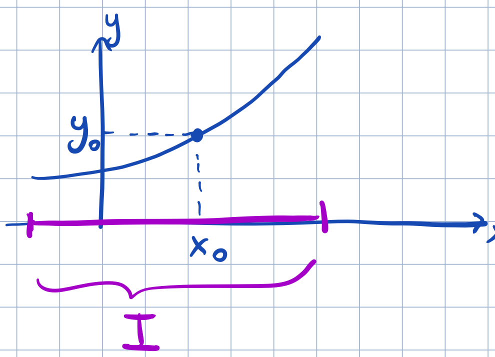
$$\left.\begin{aligned} y(0)&=3 \\ y'(\pi)&=7 \end{aligned}\right\} \text{ side conditions} \Rightarrow \begin{aligned} y(0)&=C_2=3 \\ y'(\pi)&=-C_1=7 \end{aligned} \Rightarrow \text{particular solution: } y=-7\sin x+3\cos x$$

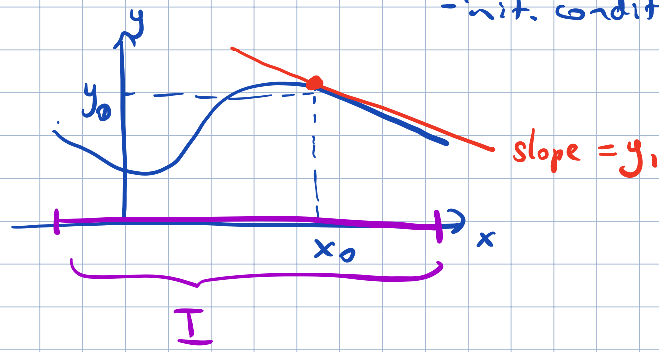
$y'(x)=C_1\cos x-C_2\sin x$

Initial Value Problems (IVP)

1st order ODE : $\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \leftarrow \text{initial condition} \end{cases}$

2nd order ODE $\begin{cases} \dfrac{d^2y}{dx^2} = f(x,y,y') \\ y(x_0) = y_0 ,\ y'(x_0) = y_1 \end{cases}$
- init. conditions





Ex: $\begin{cases} \dfrac{dy}{dx} = y \\ y(0) = 3 \end{cases}$ $\rightsquigarrow y(x) = Ce^{x}$ - general sol.
$\longrightarrow y(0) = Ce^{0} = 3 \Rightarrow y = 3e^{x}$ sol. of the IVP

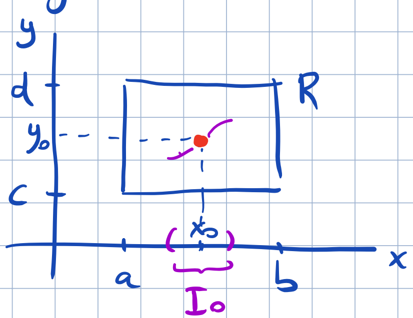
Thm (existence and uniqueness of solutions)

Consider the IVP $\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$ If there exists a rectangular region

$R: \ a < x < b ,\ c < y < d$ on the xy plane s.t.

- $(x_0, y_0) \in R$
- $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval

$I_0 = (x_0 - h, x_0 + h)$, with $h > 0$, contained in (a,b) and a unique function $y(x)$ on I_0 that is a solution of the IVP.



Remark If only f is continuous on R but $\dfrac{\partial f}{\partial y}$ is not, then we have existence of a solution, but no uniqueness.