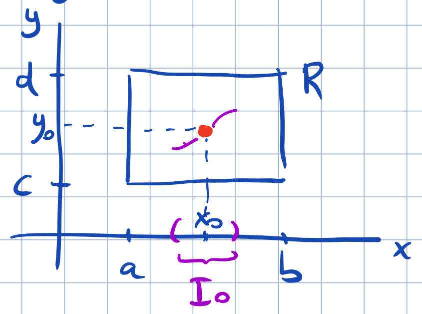
Thm (existence and uniqueness of solutions)

Consider the IVP $\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$ If there exists a rectangular region

R: $a<x<b$, $c<y<d$ on the xy plane s.t.

- $(x_0, y_0) \in R$
- $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval

$I_0 = (x_0 - h, x_0 + h)$, with $h>0$, contained in (a,b) and a unique function $y(x)$ on I_0 that is a solution of the IVP.



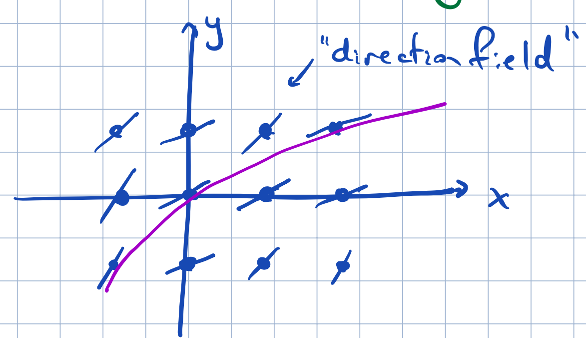
Remark If only f is continuous on R but $\dfrac{\partial f}{\partial y}$ is not, then we have existence of a solution, but no uniqueness.

Direction fields

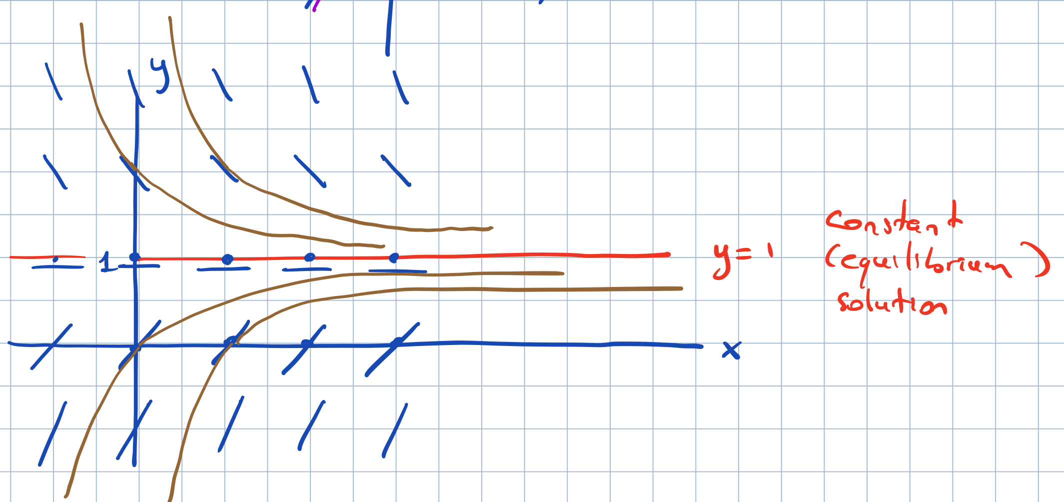
(a qualitative method for sketching solutions)

$$\frac{dy}{dx} = f(x,y)$$

$\Rightarrow$ solution curve through (x,y) must have slope $f(x,y)$



Ex: $\dfrac{dy}{dx} = 1-y$



- if we want to impose init. cond. $y(0)=2$, we should look for a curve along the direction field, passing through $(0,2)$

Autonomous equations

Autonomous 1^{st} order ODE : $\frac{dy}{dx} = f(y)$ $(*)$

no dependence on x !

Ex: $\frac{dy}{dx} = 1-y^2$ autonomous, $\frac{dy}{dx} = xy$ non-autonomous

- if c is a zero of $f(y)$, it is a "critical point" of $(*)$.

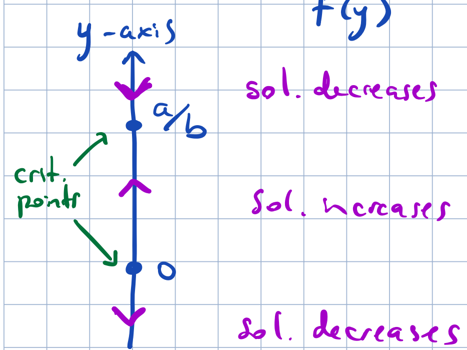
Then $y(x) = c$ is a constant (equilibrium) solution.

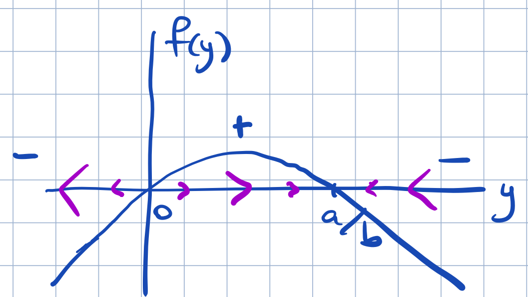
if $f(y) > 0$, solution through (x,y) is increasing

if $f(y) < 0$, —— " —— decreasing

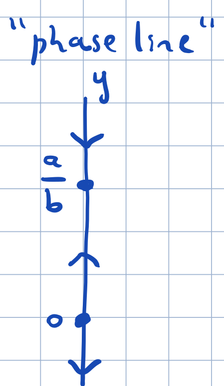
Ex: $\frac{dy}{dx} = \underbrace{y(a-by)}_{f(y)}$ - autonomous eq. Crit. points : $y=0$, $y = \frac{a}{b}$

$a, b > 0$ constants

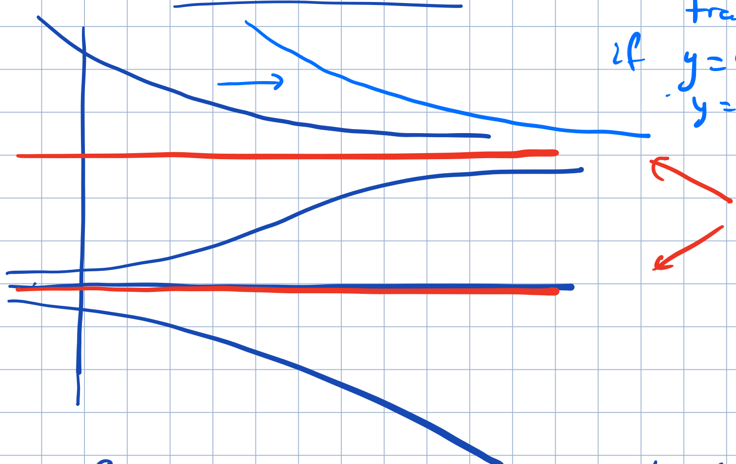




"phase line"



solution curves:



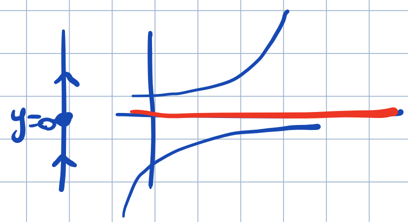
constant solutions

"translation property":
if $y = \varphi(x)$ is a sol., then $y = \varphi(x-k)$ is also a sol.
(for an autonomous ODE)

- crit. point $y = \frac{a}{b}$ is "asymptotically stable" (a sol. starting with y_0 near $\frac{a}{b}$ converges to $\frac{a}{b}$)
- crit. point $y = 0$ is "asymptotically unstable" (repels solutions)

Rem one can also have a "semi-stable" crit. point in an autonomous ODE: solutions are attracted to $y=c$ for $y_0>c$ and repelled for $y_0<c$ (or vice versa)

Ex: $\dfrac{dy}{dx}=y^2$



Separable equations (Zill 2.2)

1st order ODE $\dfrac{dy}{dx}=g(x)\,h(y)$ is said to be <u>separable</u>.

Ex: $\dfrac{dy}{dx}=e^x y^2 \rightsquigarrow \dfrac{dy}{y^2}=e^x\,dx \xrightarrow[\text{l.h.s. and r.h.s.}]{\text{integrate}} \int\dfrac{dy}{y^2}=\int e^x\,dx$

$\longrightarrow -y^{-1}+C_1=e^x+C_2$

or: $\boxed{-y^{-1}=e^x+\underbrace{C}_{\text{arbitrary constant}}}$ (#) a family of implicit solutions.

Rem: when dividing by y^2, we implicitly assumed $y\neq 0$.
actually, $y=0$ is also a (constant) solution, not a part of the family (#).

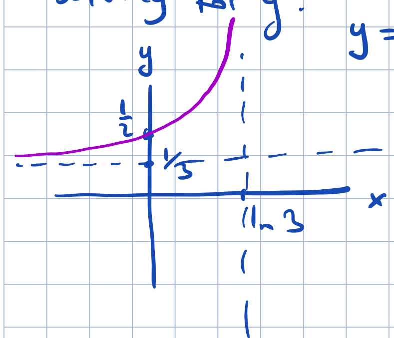
Ex: IVP $\begin{cases}\dfrac{dy}{dx}=e^x y^2\\ y(0)=\frac{1}{2}\end{cases} \Rightarrow$ $-y^{-1}=e^x+C$ substitute $x=0,\ y=-\frac{1}{2}$

$-2=e^0+C \Rightarrow C=-3$

$\Rightarrow$ implicit sol. of IVP: $\boxed{-y^{-1}=e^x-3}$

Solving for y: $y=-\dfrac{1}{e^x-3}$ - explicit sol. of IVP

interval of existence: $(-\infty,\ln 3)$



up to solutions $y=c$ with $h(c)=0$

Generally: $\dfrac{dy}{dx} = g(x)\,h(y) \iff \dfrac{dy}{h(y)} = g(x)\,dx \iff \boxed{H(y) = G(x) + C}$

$\int \dfrac{dy}{h(y)}$ $\quad \int g(x)\,dx$

Linear 1st order ODEs (method of integrating factors)

Ex: • $\dfrac{dy}{dx} = \sin x \xrightarrow[\text{in } x]{\text{integrate}} y = -\cos x + C$ ← arbitrary constant

• $\underbrace{x^2\dfrac{dy}{dx} + 2xy}_{\frac{d}{dx}(x^2 y)} = x^5 \xrightarrow[\text{in } x]{\text{integrate}} x^2 y = \dfrac{x^6}{6} + C \to y = \dfrac{x^4}{6} + \dfrac{C}{x^2}$

• $\dfrac{dy}{dx} + \dfrac{2}{x}y = x^3$ - equivalent to (by multiplying by $\mu(x) = x^2$).

General 1st order linear ODE

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \xrightarrow[\cdot\frac{1}{a_1(x)}]{} \frac{dy}{dx} + P(x)y = f(x) \quad (**)$$

- "standard form"

Idea: multiply $(**)$ by some $\mu(x)$ - "integrating factor":

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}_{\text{want it to be } \frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu'(x)y} = \mu(x)f(x)$$

- this is true if $\mu'(x) = \mu(x)P(x)$

i.e. $\dfrac{d\mu}{\mu} = P(x)\,dx$

$$\to \ln|\mu(x)| = \int P(x)\,dx + C_1 \Rightarrow \mu(x) = C_2 e^{\int P(x)\,dx}$$

· we don't need the most general integrating factor and can just take $C_2 = 1$:

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

integrating factor

So, (**) becomes $\frac{d}{dx}(\mu(x)y) = \mu(x) f(x) \xrightarrow[\text{in } x]{\text{integrate}} \mu(x) y = \int \mu(x) f(x)\,dx + C$

Thus: $\boxed{y = \frac{1}{\mu(x)}\left(\int \mu(x) f(x)\,dx + C\right)}$ - general solution of (*).

arbitrary constant

<u>Ex:</u> $\frac{dy}{dx} + \underbrace{2}_{P}y = \underbrace{5}_{f} \rightsquigarrow \underbrace{\mu = e^{\int 2\,dx}}_{\text{int. factor}} = e^{2x}, \quad y = e^{-2x}\Big(\underbrace{\int 5e^{2x}dx}_{\frac{5}{2}e^{2x}} + C\Big) = \frac{5}{2} + Ce^{-2x}$

<u>Ex:</u> $x\frac{dy}{dx} + 3y = x^{-2}e^{x} \xrightarrow[\cdot \frac{1}{x}]{} \frac{dy}{dx} + \underbrace{\boxed{\frac{3}{x}}}_{P} y = \underbrace{x^{-3}e^{x}}_{f}$ - standard form

$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = |x|^3$ consider $x > 0 \rightsquigarrow \mu = x^3$

$y = \frac{1}{x^3}\Big(\underbrace{\int \cancel{x^3}\,\cancel{x^{-3}}e^{x}dx}_{e^x} + C\Big) = \frac{e^x}{x^3} + \frac{C}{x^3}$

<u>Ex:</u> IVP $\begin{cases} x\frac{dy}{dx} + 3y = x^{-2}e^{x} \\ y(1) = 0 \end{cases}$ $\longrightarrow y = x^{-3}(e^x + C)$

$\longrightarrow 1^{-3}(e^1 + C) = 0 \rightarrow C = -e \rightarrow \boxed{y = x^{-3}(e^x - e)}$

Sol. of IVP