LAST TIME

Separable equations (Zill 2.2)

1st order ODE $\frac{dy}{dx} = g(x)\,h(y)$ is said to be separable.

Ex: $\frac{dy}{dx} = e^x y^2 \rightsquigarrow \frac{dy}{y^2} = e^x dx \xrightarrow[\text{l.h.s. and r.h.s.}]{\text{integrate}} \int \frac{dy}{y^2} = \int e^x dx$

$\longrightarrow -y^{-1} + C_1 = e^x + C_2$

or: $\boxed{-y^{-1} = e^x + \underbrace{C}_{\text{arbitrary constant}}}$ (#) a family of implicit solutions.

Rem: when dividing by y^2, we implicitly assumed $y \neq 0$.
actually, $y=0$ is also a (constant) solution, not a part of the family (#)

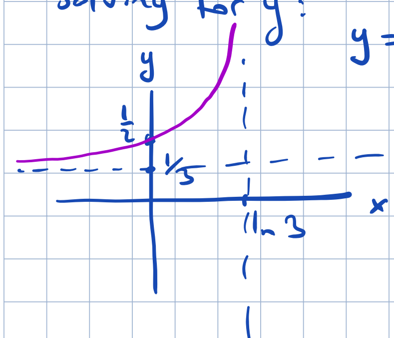
Ex: IVP $\begin{cases} \frac{dy}{dx} = e^x y^2 \\ y(0) = \frac{1}{2} \end{cases}$ $\Rightarrow$ $-y^{-1} = e^x + C$ substitute $x=0, y=-\frac{1}{2}$

$-2 = e^0 + C \Rightarrow C = -3$

$\Rightarrow$ implicit sol. of IVP: $\boxed{-y^{-1} = e^x - 3}$

solving for y: $y = -\frac{1}{e^x - 3}$ - explicit sol. of IVP

interval of existence: $(-\infty, \ln 3)$



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up to solutions $y=c$ with $h(c)=0$

<u>Generally</u>: $\frac{dy}{dx} = g(x)h(y) \iff \frac{dy}{h(y)} = g(x)\,dx \iff \boxed{H(y) = G(x) + C}$

$\int \frac{dy}{h(y)}$ $\quad\int g(x)\,dx$

<u>Linear 1st order ODEs</u> (method of integrating factors)

<u>Ex</u>:

- $\frac{dy}{dx} = \sin x \xrightarrow[\text{integrate in } x]{} y = -\cos x + C$ ← arbitrary constant
- $\underbrace{x^2\frac{dy}{dx} + 2xy}_{\frac{d}{dx}(x^2 y)} = x^5 \xrightarrow[\text{integrate in } x]{} x^2 y = \frac{x^6}{6} + C \rightarrow y = \frac{x^4}{6} + \frac{C}{x^2}$
- $\frac{dy}{dx} + \frac{2}{x}y = x^3$ - equivalent to (by multiplying by $\mu(x) = x^2$).

General 1st order linear ODE

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \xrightarrow[\cdot\frac{1}{a_1(x)}]{} \frac{dy}{dx} + P(x)y = f(x) \quad (**)$$

- "standard form"

Idea: multiply $(**)$ by some $\mu(x)$ - "integrating factor":

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}_{\text{want it to be } \frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu'(x)y} = \mu(x)f(x)$$

- this is true if $\mu'(x) = \mu(x)P(x)$

i.e. $\frac{d\mu}{\mu} = P(x)\,dx$

$$\rightarrow \ln|\mu(x)| = \int P(x)\,dx + C_1 \Rightarrow \mu(x) = C_2 e^{\int P(x)\,dx}$$

- we don't need the most general integrating factor and can just take $C_2 = 1$:

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

integrating factor

So, (**) becomes $\frac{d}{dx}(\mu(x)y) = \mu(x)f(x) \underset{\text{integrate in } x}{\rightsquigarrow} \mu(x)y = \int \mu(x)f(x)\,dx + C$

Thus: $\boxed{y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)\,dx + C\right)}$ - general solution of (*).

arbitrary constant

Ex: $\underbrace{\frac{dy}{dx} + 2}_{P}y = \underbrace{5}_{f} \rightsquigarrow \underbrace{\mu = e^{\int 2\,dx}}_{\text{int. factor}} = e^{2x}$, $y = e^{-2x}\Big(\underbrace{\int 5e^{2x}dx}_{\frac{5}{2}e^{2x}} + C\Big) = \frac{5}{2} + Ce^{-2x}$

Ex: $x\frac{dy}{dx} + 3y = x^{-2}e^x \underset{\cdot\frac{1}{x}}{\rightsquigarrow} \frac{dy}{dx} + \underbrace{\boxed{\frac{3}{x}}}_{P}y = \underbrace{x^{-3}e^x}_{f}$ - standard form

$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = |x|^3$ consider $x > 0 \rightsquigarrow \mu = x^3$

$y = \frac{1}{x^3}\Big(\underbrace{\int \cancel{x^3 x^{-3}} e^x\,dx}_{e^x} + C\Big) = \frac{e^x}{x^3} + \frac{C}{x^3}$

Ex: IVP $\begin{cases} x\frac{dy}{dx} + 3y = x^{-2}e^x \\ y(1) = 0 \end{cases}$ $\rightsquigarrow y = x^{-3}(e^x + C)$

$\rightarrow 1^{-3}(e^1 + C) = 0 \rightarrow C = -e \rightarrow \boxed{y = x^{-3}(e^x - e)}$

Sol. of IVP

Exact equations

- 1^{st} order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ (*) is "exact" if

there exists $f(x,y)$ such that $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$.

In that case,

$(*) \iff \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}_{\text{increment of } f} = 0 \rightsquigarrow$ implicit solution: $\boxed{f(x,y) = C}$

as we shift from (x,y) to $(x+dx, y+dy)$

If $(*)$ is exact, then $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial N}{\partial x}$.

So: $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$ - criterion of exactness

How to find f?

(1) $\dfrac{\partial f}{\partial x} = M(x,y) \xrightarrow[\text{in } x]{\text{integrate}} f = \int M(x,y)\,dx + \underbrace{g(y)}_{\text{"constant" of integration}}$

(2) $\dfrac{\partial f}{\partial y} = N(x,y) \longrightarrow$ substitute f into (2) and find g from it.

Ex:

$$\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$$

check: $\dfrac{\partial M}{\partial y} = 2x = \dfrac{\partial N}{\partial x}$ ✓

$\Rightarrow$ eq. is exact.

(1) $\dfrac{\partial f}{\partial x} = 2xy \xrightarrow[x]{\text{int. in}} f(x,y) = x^2y + g(y)$

(2) $\dfrac{\partial f}{\partial y} = x^2-1 \longrightarrow \dfrac{\partial f}{\partial y} = x^2 + g'(y) = x^2-1 \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y$

So: $f(x,y) = x^2y - y \quad \Rightarrow$ implicit solution: $\boxed{x^2y - y = C}$

$\rightarrow y = \dfrac{C}{x^2-1}$ explicit sol. (on any interval not containing $x=1$, $x=-1$)

Integrating factors

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, we can try to make it exact by multiplying by an integrating factor $\mu(x,y)$:

$$\boxed{\underbrace{\mu M}_{\tilde{M}}\,dx + \underbrace{\mu N}_{\tilde{N}}\,dy = 0}$$

We want $\underbrace{\dfrac{\partial \tilde{M}}{\partial y}}_{\mu M_y + \mu_y M} = \underbrace{\dfrac{\partial \tilde{N}}{\partial x}}_{\mu N_x + \mu_x N}$

$\Rightarrow \mu_x N - \mu_y M = \mu(M_y - N_x)$ (#) - complicated PDE on μ!

Case $\mu = \mu(x)$. Then (#) is: $\mu_x N = \mu(M_y - N_x)$

$\Leftrightarrow \boxed{\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}}$. if the rhs depends only on x, we can solve for $\mu(x)$

Similarly, case $\mu = \mu(y)$: $\frac{\mu_y}{\mu} = \underbrace{-\frac{M_y - N_x}{M}}_{\text{if this depends on } y \text{ only, can solve for } \mu(y).}$

Ex: $\underbrace{xy}_{M}\,dx + \underbrace{(2x^2+3y^2)}_{N}\,dy = 0$ (**) $\begin{array}{l} M_y = x \\ N_x = 4x \end{array}$ $\not=$ $\Rightarrow$ eq. not exact!

. $\frac{M_y - N_x}{N} = \frac{-3x}{2x^2+3y^2}$ depends on y (not on x only) $\Rightarrow$ cannot find $\mu(x)$.

. $-\frac{M_y - N_x}{M} = \frac{3x}{xy} = \frac{3}{y}$ depends on y only $\Rightarrow$ can find $\mu(y)$, $\frac{\mu_y}{\mu} = \frac{3}{y}$

$\rightarrow \mu = e^{3\ln y} = y^3$ $\rightarrow y^3 \cdot (**)$: $\underbrace{xy^4}_{\tilde{M}}\,dx + \underbrace{(2x^2y^3 + 3y^5)}_{\tilde{N}}\,dy = 0$ - exact

(1) $f_x = xy^4 \underset{\text{int in } x}{\rightsquigarrow} f = \frac{x^2y^4}{2} + g(y)$

(2) $f_y = 2x^2y^3 + 3y^5 \rightsquigarrow f_y = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 \rightarrow g'(y) = 3y^5$

$\rightarrow g(y) = \frac{1}{2}y^6$

$\Rightarrow f = \frac{x^2y^4}{2} + \frac{y^6}{2}$

$\Rightarrow \boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 = C}$ - implicit solution.