Variation of parameters (Zill 4.6)

2^{nd} order linear nonhomog. ODE $\quad y'' + P(x)y' + Q(x)y = f(x) \qquad (*)$

assume we know a FSS y_1, y_2 for the associated homog. eq.

$$y'' + P(x)y' + Q(x)\,y = 0 \qquad (**)$$

i.e. the "complementary function" is $y_c = c_1 y_1(x) + c_2 y_2(x)$

Idea: look for a particular sol. of $(*)$ of the form

$$Y = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

"variable parameters"
- functions to be determined

Idea (Lagrange): To get two equations on u_1, u_2, impose an "auxiliary equation":

$$\boxed{u_1' y_1 + u_2' y_2 = 0} \qquad (\#)$$

Substitute Y into $(*)$:

$$Q\cdot \mid Y = u_1 y_1 + u_2 y_2$$

$$P\cdot \mid Y' = u_1 y_1' + u_2 y_2' + \cancel{u_1' y_1 + u_2' y_2}$$

$$1\cdot \mid Y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$$Y'' + PY' + QY = u_1 \underbrace{(y_1'' + Py_1' + Qy_1)}_{0 \text{ by homog. eq.}} + u_2 \underbrace{(y_2'' + Py_2' + Qy_2)}_{0} + u_1' y_1' + u_2' y_2' \overset{\text{WANT}}{=} f$$

So, we have two equations

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 & \leftarrow \text{aux. eq. } (\#) \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

- linear system on $u_1'(x), u_2'(x)$

From Cramer's rule: $\quad u_1' = \dfrac{W_1}{W} = \dfrac{-y_2 f}{W} \quad (1) \qquad u_2' = \dfrac{W_2}{W} = \dfrac{y_1 f}{W} \quad (2)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = -y_2 f \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = y_1 f$$

Wronskian

$\rightarrow$ find u_1, u_2 by integrating (1), (2) $\rightarrow$ part. sol.: $Y = u_1 y_1 + u_2 y_2$

Thus: $$\boxed{Y = y_1(x)\int \frac{-y_2(x) f(x)}{W(y_1,y_2)(x)}\,dx + y_2(x)\int \frac{y_1(x) f(x)}{W(y_1,y_2)(x)}\,dx}$$ - part. sol. of (*)

Ex: $y'' - 4y' + 4y = (x+1)e^{2x}$ find the gen. sol.

Sol: $y_1 = e^{2x}, y_2 = xe^{2x}$ - FSS of $y'' - 4y' + 4y = 0$

$$W(y_1,y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{vmatrix} = e^{4x} \rightsquigarrow$$

$\rightsquigarrow$ (1) $u_1' = \frac{-xe^{2x}\cdot(x+1)e^{2x}}{e^{4x}} = -x^2 - x \xrightarrow{\text{integrate}} u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$

(2) $u_2' = \frac{e^{2x}\cdot(x+1)e^{2x}}{e^{4x}} = x+1 \rightsquigarrow u_2 = \frac{x^2}{2} + x$

So: $$Y = \underbrace{\left(-\frac{x^3}{3} - \frac{x^2}{2}\right)}_{u_1}\underbrace{e^{2x}}_{y_1} + \underbrace{\left(\frac{x^2}{2} + x\right)}_{u_2}\underbrace{xe^{2x}}_{y_2} = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$ - part. sol.

$$y = y_c + Y = c_1e^{2x} + c_2xe^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$ - gen. sol.

Ex: $y'' + y = \frac{1}{\cos x}$ on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Sol: $y_c = c_1\underbrace{\cos x}_{y_1} + c_2\underbrace{\sin x}_{y_2}$ $\quad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

(1) $u_1' = \frac{-\frac{\sin x}{\cos x}}{1} = -\frac{\sin x}{\cos x} \xrightarrow{\text{int.}} u_1 = \ln \cos x$

(2) $u_2' = \frac{\frac{\cos x}{\cos x}}{1} = 1 \xrightarrow{\text{int.}} u_2 = x$

$\Rightarrow Y(x) = \ln(\cos x)\cos x + x\sin x$

$y = c_1\cos x + c_2\sin x + \ln(\cos x)\cos x + x\sin x$ - gen. sol.

Green's function (Zill 4.8.1)

Variation of parameters $\Rightarrow$ $Y(x) = y_1(x)\displaystyle\int_{x_0}^{x} \frac{-y_2(t) f(t)}{W(t)}\,dt + y_2(x) \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\,dt$ – part. sol. of (*)

$$\Rightarrow Y(x) = \int_{x_0}^{x} G(x,t) f(t)\,dt$$

with $\boxed{G(x,t) = \dfrac{-y_1(x) y_2(t) + y_2(x) y_1(t)}{W(t)}}$ – "Green's function" for (*) (or (**))

Thm (a) $Y(x) = \displaystyle\int_{x_0}^{x} G(x,t) f(t)\,dt$ is a solution of the IVP $\begin{cases} y'' + P(x) y' + Q(x) y = f(x) \\ \boxed{y(x_0) = 0,\ y'(x_0) = 0} \end{cases}$

"homog. initial condition"

(b) $G(x,t)$ is a sol. of the IVP $\begin{cases} y'' + P(x) y' + Q(x) y = 0 \\ y(t) = 0,\ y'(t) = 1 \end{cases}$

Ex: $y'' - y = \underbrace{f(x)}_{\text{some function}}$ – write a particular sol. $Y(x)$

Sol: homog. eq. $y'' - y = 0$ $\rightsquigarrow$ $y_1 = e^{x}$, $y_2 = e^{-x}$ – FSS; $W(y_1, y_2) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$

$m^2 - 1 = 0$

$$\Rightarrow G(x,t) = \frac{-y_1(x) y_2(t) + y_2(x) y_1(t)}{W(t)} = \frac{-e^{x} e^{-t} + e^{-x} e^{t}}{-2} = \frac{e^{x-t} - e^{t-x}}{2} = \sinh(x-t)$$

Thus: $Y(x) = \displaystyle\int_{x_0}^{x} \sinh(x-t) f(t)\,dt$ – a part. sol. of

Ex: $\begin{cases} y''-y=e^{2x} \\ y(0)=0,\ y'(0)=0 \end{cases}$ - solve the IVP

Sol: $Y(x)=\int_{\boxed{0}}^{x} \underbrace{\sinh(x-t)}_{G(x,t)}\,\underbrace{e^{2t}}_{f(t)}\,dt = \int_0^x \frac{1}{2}\left(e^{x+t}-e^{3t-x}\right)dt = \frac{1}{2}\left(e^x(e^x-1)-e^{-x}\frac{1}{3}(e^{3x}-1)\right)$

$= \frac{1}{3}e^{2x}-\frac{1}{2}e^{x}+\frac{1}{6}e^{-x}$

x_0 where init. cond. is given

gen. sol: $y(x)= c_1e^x+c_2e^{-x}+\left(\frac{1}{3}e^{2x}-\frac{1}{2}e^{x}+\frac{1}{6}e^{-x}\right)$

Thm The sol. of IVP $\begin{cases} y''+Py'+Qy=f \\ y(x_0)=y_0,\ y'(x_0)=y_1 \end{cases}$ on an interval I

can be obtained as $y(x)=\underbrace{y_h(x)}_{\text{response of the system to init. conditions}}+\underbrace{Y(x)}_{\text{response of the sys. to the "forcing function" } f(x)}$ with

y_h a sol. of the IVP $\begin{cases} y''+Py'+Qy=0 \\ y(x_0)=\underline{y_0},\ y'(x_0)=\underline{y_1} \end{cases}$ and Y a sol. of IVP $\begin{cases} y''+Py'+Qy=f \\ y(x_0)=0,\ y'(x_0)=0 \end{cases}$

Ex: Solve the IVP $\begin{cases} y''+4y=\sin 2x \\ y(0)=1,\ y'(0)=-2 \end{cases}$

Sol: (a) homog. eq.
$y''+4y=0$
$m^2+4=0$
$\rightarrow y_1=\cos 2x,\ y_2=\sin 2x$ - FSS $\qquad W=\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}=2$

$G(x,t)=\dfrac{-\cos 2x\cdot\sin 2t+\sin 2x\cos 2t}{2}=\dfrac{\sin 2(x-t)}{2}$ Green's function

$Y(x)=\int_0^x \frac{\sin 2(x-t)}{2}\sin 2t\,dt = \int_0^x \frac{1}{4}\left(\cos(2x-4t)-\cos 2x\right)dt$

$\sin\alpha\cdot\sin\beta=\dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

$= \left|-\frac{1}{16}\sin(2x-4t)\right|_{t=0}^{t=x} - \frac{1}{4}x\cos 2x$

$= \frac{1}{8}\sin 2x-\frac{1}{4}x\cos 2x$ part. sol. of NH eq. with $y(0)=0,\ y'(0)=0$

(b) $\begin{cases} y'' + 4y = 0 \\ y(0) = 1,\ y'(0) = -2 \end{cases} \to y = c_1 \cos 2x + c_2 \sin 2x \to y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$y(0) = c_1 = 1$

$y'(0) = 2c_2 = -2$ $\Rightarrow$ $c_1 = 1$, $c_2 = -1$

$\Rightarrow y_h = \cos 2x - \sin 2x$

$\Rightarrow y = y_h + Y = \boxed{\cos 2x - \frac{7}{8} \sin 2x - \frac{1}{4} x \cos 2x}$ sol. of original IVP

Ex: $\begin{cases} y'' + 4y = x \\ y'(0) = 1,\ y'(0) = -2 \end{cases}$

Sol: same y_h, same G

$$Y = \int_0^x \underbrace{\frac{\sin 2(x-t)}{2}}_{G(x,t)} \underbrace{t}_{f(t)}\, dt = \int_0^x \left(\frac{1}{4} \cos 2(x-t)\right)' t\, dt \underset{\text{int. by parts}}{=} \left|\frac{1}{4} \cos 2(x-t) \cdot t\right|_0^x - \int_0^x \frac{1}{4} \cos 2(x-t)\, dt$$

$$= \frac{x}{4} - \left|\frac{-1}{8} \sin 2(x-t)\right|_0^x = \frac{x}{4} - \frac{1}{8} \sin 2x.$$

$\Rightarrow y = \underbrace{y_h}_{\text{as before}} + Y = \boxed{\frac{x}{4} - \frac{9}{8} \sin 2x + \cos 2x}$