LAST TIME

$$ay'' + by' + cy = g(x)$$

Ex: $y'' - 3y' - 4y = 3e^{2x}$ (#) find Y

Sol: try looking for Y of the form $Y = Ae^{2x}$ ← coeff. to be determined $\to Y' = 2Ae^{2x}$, $Y'' = 4Ae^{2x}$

$\Rightarrow Y'' - 3Y' - 4Y = \underbrace{(4A - 6A - 4A)}_{-6A} e^{2x} \overset{\text{WANT}}{=} 3e^{2x} \Rightarrow A = -\frac{1}{2}$

$\Rightarrow \boxed{Y = -\frac{1}{2}e^{2x}}$ a solution

- complementary function: $y_c'' - 3y_c' - 4y_c = 0 \rightsquigarrow$ aux eq. $m^2 - 3m - 4 = 0$

$\to m_1 = 4,\ m_2 = -1$

$\rightsquigarrow y_c = c_1 e^{4x} + c_2 e^{-x}$ - complementary function

$\rightsquigarrow$ gen. sol of (#): $y = y_c + Y = \boxed{c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}}$

Ex: $y'' - 3y' - 4y = 2\sin x$

Sol: try $Y = A\sin x \rightsquigarrow Y' = A\cos x$, $Y'' = -A\sin x$

$\rightsquigarrow Y'' - 3Y' - 4Y = \underbrace{(-A - 4A)}_{-5A}\sin x - 3A\cos x \overset{\text{WANT}}{=} 2\sin x$

$x = 0 \to -3A = 0$
$x = \frac{\pi}{2} \to -5A = 2$
} inconsistent! $\Rightarrow$ cannot find A s.t. $Y = A\sin x$ is a sol.

<u>Next try:</u> $Y = A\sin x + B\cos x$

$$\Rightarrow \quad Y' = A\cos x - B\sin x$$
$$Y'' = -A\sin x - B\cos x$$

$$Y'' - 3Y' - 4Y = \underbrace{(-A + 3B - 4A)}_{-5A+3B}\sin x + \underbrace{(-B - 3A - 4B)}_{-3A-5B}\cos x \overset{\text{WANT}}{=} 2\sin x$$

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases} \Rightarrow \begin{cases} A = -5/17 \\ B = 3/17 \end{cases} \Rightarrow Y = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$$

- for $g(x)$ a polynomial, try Y a polynomial of the same degree.

<u>Ex:</u> $y'' - 3y' - 4y = 4x^2 - 1 \quad \rightsquigarrow$ try $Y = Ax^2 + Bx + C$

$$\Rightarrow \quad Y' = 2Ax + B$$
$$Y'' = 2A$$

So, $Y'' - 3Y' - 4Y = -4Ax^2 + (-6A - 4B)x + (2A - 3B - 4C) \underset{\text{WANT}}{=} 4x^2 - 1$

$$\begin{cases} -4A = 4 \\ -6A - 4B = 0 \\ 2A - 3B - 4C = -1 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 3/2 \\ C = -11/8 \end{cases} \Rightarrow Y = -x^2 + \frac{3}{2}x - \frac{11}{8}$$

Thus:
- for $g(x) = e^{\alpha x}$, try $Y = Ae^{\alpha x}$
- for $g(x) = \sin\beta x$, try $Y = A\sin\beta x + B\cos\beta x$
- for $g(x)$ a polynomial of degree n, try Y a poly. of degree n.

- for g a product of two or three of these types of functions, take a product of these forms of Y

Ex: $y'' - 3y' - 4y = -8e^{x}\cos 2x$

Sol: Try $Y = Ae^{x}\cos 2x + Be^{x}\sin 2x$

$$Y' = (A+2B)e^{x}\cos 2x + (B-2A)e^{x}\sin 2x$$

$$Y'' = \underbrace{(A+2B+2B-4A)}_{-3A+4B}e^{x}\cos 2x + \underbrace{(B-2A-2A-4B)}_{-4A-3B}e^{x}\sin 2x$$

$$Y'' - 3Y' - 4Y = \Big(\underbrace{(-3A+4B)-3(A+2B)-4A}_{-10A-2B}\Big)e^{x}\cos 2x + \Big(\underbrace{(-4A-3B)-3(-2A+B)-4B}_{2A-10B}\Big)e^{x}\sin 2x \overset{\text{WANT}}{=} -8e^{x}\cos 2x$$

$$\begin{cases} -10A - 2B = -8 \\ 2A - 10B = 0 \end{cases} \longrightarrow \begin{array}{l} A = \frac{10}{13} \\ B = \frac{2}{13} \end{array} \Rightarrow Y = \frac{10}{13}e^{x}\cos 2x + \frac{2}{13}e^{x}\sin 2x$$

- If $g(x) = g_1(x) + g_2(x)$, Y_1, Y_2 are solutions of $ay'' + by' + cy = g_1(x)$, $ay'' + by' + cy = g_2(x)$

then $Y = Y_1 + Y_2$ is a sol. of $ay'' + by' + cy = g(x)$

Ex: $y'' - 3y' - 4y = 3e^{2x} + 2\sin x$

Sol: $y'' - 3y' - 4y = 3e^{2x} \longrightarrow Y_1 = -\frac{1}{2}e^{2x}$

$y'' - 3y' - 4y = 2\sin x \longrightarrow Y_2 = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$

$\Rightarrow Y = Y_1 + Y_2 = -\frac{1}{2}e^{2x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x$ - a solution.

Ex [issue]: $y'' - 3y' - 4y = 2e^{-x}$

try $Y = Ae^{-x} \rightarrow Y' = -Ae^{-x}$, $Y'' = Ae^{-x}$

$\Rightarrow Y'' - 3Y' - 4Y = \underbrace{(A+3A-4A)}_{0}e^{-x} \overset{\text{WANT}}{=} 2e^{-x}$

- does not work!

- because e^{-x} is a sol. of the homog. eq.

Next try: $Y = A\,x\,e^{-x} \rightarrow Y' = A(1-x)e^{-x}$, $Y'' = A(x-2)e^{-x}$

$$\Rightarrow Y'' - 3Y' - 4Y = A\underbrace{\left((x-2) - 3(1-x) - 4x\right)}_{-5}e^{-x} = 2e^{-x}$$

$$\Rightarrow A = -\frac{2}{5}, \quad Y = -\frac{2}{5}xe^{-x}$$

- If a term in the assumed form of the solution is itself a sol. of the assoc. homog. eq., then multiply it by x. In case it again contains a term which is a sol. of assoc. homog. eq., multiply by x a second time.

  (For 2^{nd} order ODEs, no further iterations can happen)

- For $ay'' + by' + cy = g(x)$ with $g(x) = g_1(x) + \dots + g_N(x)$

$$g_i(x) = \begin{cases} P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0 \\ P_n(x)e^{\alpha x} \\ P_n(x)e^{\alpha x}\begin{cases}\sin\beta x \\ \cos\beta x\end{cases} \end{cases}$$

$$Y_i = x^s(A_n x^n + A_{n-1}x^{n-1} + \dots + A_0)$$

$$Y_i = x^s(A_n x^n + \dots + A_0)\,e^{\alpha x}$$

$$Y_i = x^s\left((A_n x^n + \dots + A_0)e^{\alpha x}\cos\beta x + (B_n x^n + \dots + B_0)e^{\alpha x}\sin\beta x\right)$$

$s \in \{0, 1, 2\}$ smallest number s.t. no term in $Y_i(x)$ is a sol. of homog. eq.

Then: $Y = Y_1 + \dots + Y_N$

Ex: $y'' - 2y' + y = xe^x$

Sol: $y_1 = e^x$, $y_2 = xe^x$ - FSS of homog. eq. $y'' - 2y' + y = 0$

try: $Y = (Ax + B)e^x$ - both terms are sols of homog. eq. X

try 2: $Y = x(Ax + B)e^x$ - second term is a sol. of homog. eq. X try 3: $Y = x^2(Ax+B)e^x$ · no terms are sols of homg. eq. ✓

$$Y' = (Ax^3 + (B+3A)x^2 + 2Bx)e^x$$

$$Y'' = (Ax^3 + (B+6A)x^2 + (4B+6A)x + 2B)e^x$$

$$\rightsquigarrow Y'' - 2Y' + Y = (\underbrace{(A-2A+A)}_{0}x^3 + \underbrace{(6A+B-2(3A+B)+B)}_{0}x^2 + \underbrace{(6A+4B-4B)}_{6A}x + 2B)e^x \overset{\text{WANT}}{=} x e^x$$

$$\Rightarrow \begin{matrix} 6A = 1 \\ 2B = 0 \end{matrix} \Rightarrow \begin{matrix} A = 1/6 \\ B = 0 \end{matrix} \Rightarrow Y = \frac{1}{6}x^3 e^x$$

Ex: IVP
$$\begin{cases} y'' - 2y' + y = \overbrace{xe^x}^{g_1} + \overbrace{1}^{g_2} \\ y(0) = 0, \; y'(0) = 1 \end{cases} \rightsquigarrow y = \underbrace{c_1 e^x + c_2 x e^x}_{y_c} + \underbrace{\frac{1}{6}x^3 e^x}_{Y_1} + \underbrace{1}_{Y_2}$$

$$y' = c_1 e^x + c_2(x+1)e^x + \frac{1}{6}(x^3 + 3x^2)e^x$$

$$\begin{matrix} y(0) = c_1 + 1 = 0 \\ y'(0) = c_1 + c_2 = 1 \end{matrix} \Rightarrow \begin{matrix} c_1 = -1 \\ c_2 = 2 \end{matrix}$$

$$\Rightarrow y = -e^x + 2xe^x + \frac{1}{6}x^3 e^x + 1$$ – sol. of the IVP.