1. (5pts)Which vector below is an eigenvector for the matrix $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ with complex eigenvalue the one with positive imaginary part?

(a) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ (b) $\begin{bmatrix} -2.00 \\ -3i \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ -3i \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ (e) $\begin{bmatrix} 3+2i \\ 2+3i \end{bmatrix}$

$\begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + a^2+b^2$

$\lambda = a \pm ib$

$\lambda = a+ib$: $A - \lambda I = \begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix}$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$ - eigenvector with $\lambda = a+ib$

2. (5pts)The three vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ -1 \\ 5 \\ 6 \\ 1 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ -7 \\ 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal. If $\mathbf{W}$ is the subspace of $\mathbb{R}^7$ spanned by these three vectors, what is the dimension of $\mathbf{W}^\perp$?

$\dim W = 3 \Rightarrow \dim W^\perp = 7-3 = 4$

(a) 5

(b) 4

(c) 3

(d) $\mathbf{W}^\perp$ is not a subspace.

(e) Can not be determined from the given information.

3. (5pts)Which vector below is the projection of $\overset{x}{\begin{bmatrix}3\\0\\1\\0\end{bmatrix}}$ to the plane spanned by the orthogonal vectors $\overset{v_1}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}$ and $\overset{v_2}{\begin{bmatrix}2\\-1\\-4\\3\end{bmatrix}}$?

(a) $\frac{1}{3}\begin{bmatrix}2.5\\-3.5\\-9.5\\1.5\end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix}-2\\-5\\-8\\-9\end{bmatrix}$ (c) $\frac{1}{3}\begin{bmatrix}-5\\4\\13\\-6\end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix}1\\1\\1\\3\end{bmatrix}$ (e) $\frac{1}{3}\begin{bmatrix}4\\7\\10\\15\end{bmatrix}$

$$\text{proj}_W x = \frac{x\cdot v_1}{v_1\cdot v_1}v_1 + \frac{x\cdot v_2}{v_2\cdot v_2}v_2 = \frac{6}{30}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \frac{2}{30}\begin{bmatrix}2\\-1\\-4\\3\end{bmatrix} = \frac{1}{30}\begin{bmatrix}10\\10\\10\\30\end{bmatrix}$$

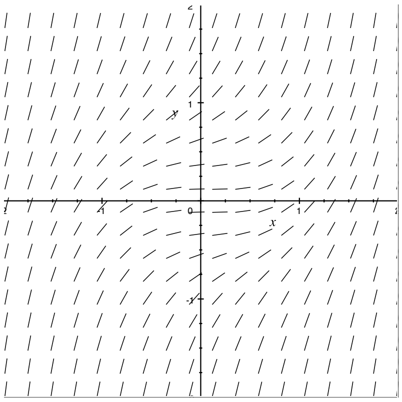
4. (5pts)Find the least squares solution to the inconsistent equation $\overset{A}{\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}}\mathbf{x} = \overset{b}{\begin{bmatrix}1\\4\\3\end{bmatrix}}$.

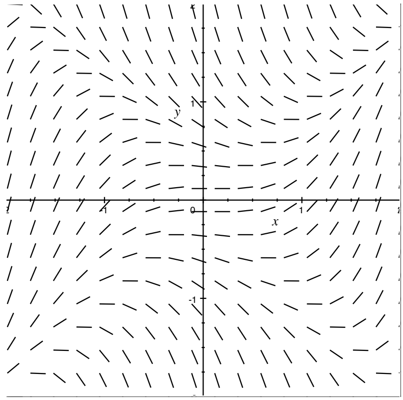
(a) $\begin{bmatrix}2\\4\end{bmatrix}$ (b) $\begin{bmatrix}2\\0\\4\end{bmatrix}$ (c) $\begin{bmatrix}6\\0\\20\end{bmatrix}$ (d) $\begin{bmatrix}-1\\5\end{bmatrix}$ (e) $\begin{bmatrix}6\\20\end{bmatrix}$

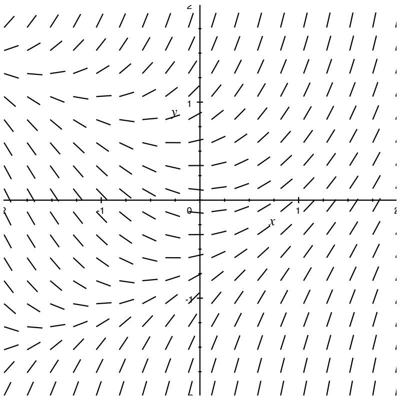
$$A^TA = \begin{bmatrix}2&0\\0&1\end{bmatrix} \qquad A^Tb = \begin{bmatrix}4\\4\end{bmatrix}$$

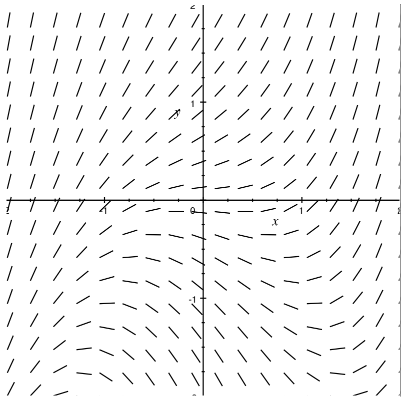
$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix}2\\4\end{bmatrix}$$

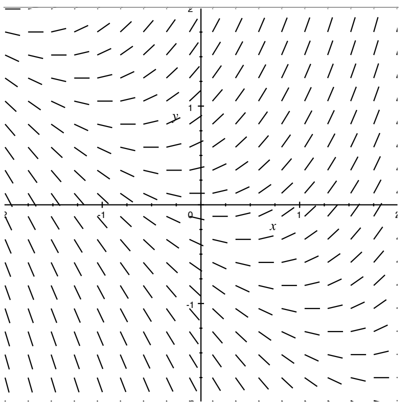
5. (5pts)Below are 5 direction fields. Which one could be the field for the equation $y' = x + y$?

(a)


(b)


(c)


(d)


(e)


6. (5pts)The differential equation $y' = x^2 y + e^x$ is

(a) linear.

(b) linear and autonomous.

(c) non-linear and autonomous.

(d) separable and linear.

(e) separable but not linear.

7. (5pts)Which function below is an integrating factor for the linear equation

$$y' = x^3 - \frac{2x+1}{x^2+x+1} y \ ?$$

P

(a) e^{x^2+x} (b) x^2+x (c) $\int_0^x \frac{s^3}{s^2+s+1}\, ds$

(d) $\frac{-2x^2-2x+1}{(x^2+x+1)^2}$ (e) x^2+x+1

$\mu = e^{\int P(x)dx} = e^{\ln(x^2+x+1)} = x^2+x+1$

8. (5pts)If $y(t)$ is the solution to the initial value problem $y' = \frac{\sin(t)}{\cos(y)}$, $y(0) = 0$, what is $y\left(\frac{\pi}{2}\right)$?

(a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$ (e) 0

$\cos y \cdot dy = \sin t \cdot dt \underset{\text{int}}{\Rightarrow} \sin y = -\cos t + C$

subst $t_0 = 0, y_0 = 0 \Rightarrow 0 = -1 + C \Rightarrow C = 1$

$\Rightarrow \sin y = 1 - \cos t$

$\underset{\sin}{\Rightarrow} y\left(\frac{\pi}{2}\right) = 1$

9. (5pts)Find all the equilibrium solutions to the autonomous equation

$$\frac{dy}{dx} = y^2 - y^4$$

and indicate which are asymptotically stable, unstable or neither.

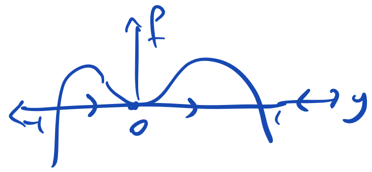
(a) -1 is unstable; 0 is asymptotically stable and 1 is asymptotically stable.

(b) -1 is neither; 0 is asymptotically stable and 1 is asymptotically stable.

(c) -1 is unstable; 0 is neither and 1 is asymptotically stable.

(d) -1 is asymptotically stable; 0 is asymptotically stable and 1 is neither.

(e) -1 is unstable; 0 is unstable and 1 is asymptotically stable.



10. (12pts)Apply the Gram-Schmidt process to the three vectors given below to get an orthogonal basis for $\mathbb{R}^3$.

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

You need not normalize your basis and you should clear fractions wherever possible.

$v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ -4/3 \end{bmatrix} \xrightarrow{\cdot \frac{3}{4}} v_2' = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \underset{1/3}{\frac{2}{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$\rightarrow v_3' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\left\{ v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, v_2' = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v_3' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

11. (12pts)Material not on the coming exam.

A tank initially contains 200 liters of brine (salt solution) with a salt concentration of 4 grams per liter. At some instant brine with a salt concentration of 0.4 grams per liter begins to flow into the tank at a rate of 3 liters per minute, while the well-stirred mixture flows out at the same rate. How long will it take for the salt concentration in the tank to be reduced to 0.7 grams per liter

12. (12pts)

(1) Solve the initial value problem $y' = \dfrac{3t^2}{2y(1+t^3)}$, $y(0) = -1$.

(2) State the interval of existence for this solution.

$$2y\,dy = \frac{3t^2}{1+t^3}\,dt \rightsquigarrow y^2 = \ln(1+t^3) + C$$

$(y_0 = -1,\ t_0 = 0$ $\qquad 1 = \cancel{\ln 1} + C \Rightarrow C = 1$

$$\Rightarrow y^2 = \ln(1+t^3) + 1$$

$$\infty > 1+t^3 > e^{-1} \qquad \Rightarrow t > (e^{-1}-1)^{\frac{1}{3}}$$

$$\Rightarrow I = \left((e^{-1}-1)^{1/3}, \infty\right)$$

Solutions

1. The characteristic equation is $\lambda^2 - 4\lambda + 13$. The roots are $\lambda = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$. The requested eigenvalue is $\lambda = 2 + 3i$. Hence we need a null vector for

$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -3i & -3.00 \\ 3.00 & -3i \end{bmatrix}$. A null vector is $\begin{bmatrix} 3 \\ -3i \end{bmatrix}$.

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -3i \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

$$(2+3i) \begin{bmatrix} 3 \\ -3i \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 9 \\ -6 \end{bmatrix} i$$

2. Since the three vectors are orthogonal, $\dim \mathbf{W} = 3$. Then $\dim \mathbf{W}^{\perp} = 7 - 3 = 4$.

3. Let $\mathbf{W}$ be the space spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Then

$$\text{proj}_{\mathbf{W}}(\mathbf{x}) = \frac{\mathbf{x} \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x} \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2}\mathbf{v}_2 = \frac{6}{30}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \frac{2}{30}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 10 \\ 10 \\ 10 \\ 30 \end{bmatrix}$$

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$. Then we need to solve the equation $\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{b}$ or

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

5. If you fix a horizontal line ($y =$ constant), the elements should get steeper as you move to the right. If you look in the 3rd quadrant in (b) and (d), the elements appear to get less steep.
If you fix a vertical line ($x =$ constant), the elements should get steeper as you move up. If you look in the 3rd quadrant in (c) and (e), the elements appear to get less steep.

6. The equation is linear. Standard form: $y' - x^2y = e^x$. It is not autonomous because $f(x, y) = x^2y + e^x$ depends on x and not just on y. It is not separable because $f(x, y)$ can not be written as a function of x times a function of y.

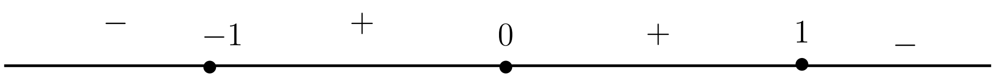
7. The equation is linear and the standard form is $y' + \dfrac{2x+1}{x^2+x+1}y = x^3$. Hence $p(x) = \dfrac{2x+1}{x^2+x+1}$ and we need to find $q(x) = \displaystyle\int \frac{2x+1}{x^2+x+1}\,dx$. The integral is a substitution (or by inspection) $q(x) = \ln|x^2+x+1| + C$ and we may take any value for C we like. Take $C = 0$. Then

$$\mu(x) = e^{\ln|x^2+x+1|} = |x^2+x+1|$$

The polynomial x^2+x+1 has no real roots so is always negative or always positive. Since it is 1 at $x = 0$ it is always positive and $\mu(x) = x^2+x+1$.

8. The equation is separable so $\cos(y)\,dy = \sin(t)\,dt$ and hence $\sin(y) = -\cos(t) + C$ and since $y(0) = 0$, $C = 1$. Hence $\sin\left(y\left(\frac{\pi}{2}\right)\right) = 1 - \cos\left(\frac{\pi}{2}\right) = 1$. Hence $y\left(\frac{\pi}{2}\right) = \dfrac{\pi}{2}$.

9. $0 = p(y) = y^2 - y^4 = y^2(1-y^2) = y^2(1-y)(1+y)$ so the equilibrium solutions are $y = 0$, $y = 1$ and $y = -1$. Then at $p(-2) < 0$, $p(-0.5) > 0$; $p(0.5) > 0$, $p(2) < 0$.



Hence -1 is unstable; 0 is neither and 1 is asymptotically stable.

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{u}_1\,\mathbf{u}_1^T = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \quad \mathbf{X}_1 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_1 = \begin{bmatrix} 8 & 14 \\ 8 & 2 \\ -8 & 10 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B}_2 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$$

$$\mathbf{u}_2\,\mathbf{u}_2^T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{bmatrix} 18 \\ 0 \\ 18 \end{bmatrix}$$

Hence

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and we have our orthogonal basis:

10.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

OR

For the first vector, take $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Then

$$\mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix}$$

so we may take

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Next

$$\mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

so we may take $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

11.

Material not on the coming exam.

Let $V(t)$ be the volume (lit) of brine in the tank at time t minutes. Let $S(t)$ be the mass (gr) of salt in the tank at time t minutes. Because the mixture is assumed to be well-stirred, the salt concentration of the brine in the tank at time t is therefore $C(t) = \frac{S(t)}{V(t)}$. In particular, this will be the concentration of the brine that flows out of the tank.

$V(0) = 200$ and $C(0) = 4$. We need to solve $C(T) = 0.7$. Since the volume is constant, $V(t) = V(0) = 200$, $S(t) = C(t) \cdot 200$. We also have $S(0) = C(0) \cdot 200 = 4 \cdot 200 = 800$. Hence we can equivalently solve $S(T) = 0.7 \cdot 200 = 140$.

$$\frac{dS}{dt} = \text{Rate in} - \text{Rate out} = 0.4 \cdot 3 - \frac{S(t)}{V(0)} \cdot 3 = 1.2 - \frac{S(t)}{200} \cdot 3$$

This equation is linear with standard form

$$\frac{dS}{dt} + \frac{3}{200}S = 1.2$$

Since $p(t) = \frac{3}{200}$, $q(t) = \frac{3}{200}t$ so $\mu = e^{\frac{3t}{200}}$ and

$$\frac{de^{\frac{3t}{200}}S}{dt} = 1.2 \cdot e^{\frac{3t}{200}}$$

Hence

$$e^{\frac{3t}{200}}S = 1.2 \cdot e^{\frac{3t}{200}} \cdot \frac{200}{3} + C = \frac{240}{3}e^{\frac{3t}{200}} + C = 80e^{\frac{3t}{200}} + C$$

$$S(t) = 80 + Ce^{-\frac{3t}{200}}$$

Since $S(0) = 800$, $80 + C = 800$, $C = 720$. so

$$S(t) = 80 + 720e^{-\frac{3t}{200}}$$

We need to solve

$$S(T) = 140 = 80 + 720e^{-\frac{3T}{200}}$$

or

$$720e^{-\frac{3T}{200}} = 60$$

or

$$e^{-\frac{3T}{200}} = \frac{60}{720} = \frac{1}{12}$$

or

$$-\frac{3T}{200} = \ln\left(\frac{1}{12}\right) = \ln(12)$$

$$T = \frac{200}{3} \cdot \ln(12)$$

12.

$F(t, y) = \dfrac{3t^2}{2y(1+t^3)}$ which is defined and continuous everywhere except for $t = -1$ or $y = 0$ or if you prefer its the plane minus the vertical line $t = -1$ and minus the t-axis.
$\dfrac{\partial F}{\partial y} = -\dfrac{3t^2}{2y^2(1+t^3)}$ so it is defined and continuous on the domain of F.
$y' = \dfrac{3t^2}{2y(1+t^3)}$, $y(0) = -1$ so $2y\,dy = \dfrac{3t^2}{1+t^3}\,dt$ $y^2 = \ln|1+t^3| + C$ so $C = 1$ and $y = \pm\sqrt{\ln|1+t^3|+1}$ and since $y(0) = -1$, $y = -\sqrt{\ln|1+t^3|+1}$. Since $t > -1$,

$$y = -\sqrt{\ln(1+t^3)+1}$$

The function y is defined as long as the quantity under the square root is non-negative. It is 0 when $1 + t^3 = e^{-1}$ so $t^3 = e^{-1} - 1$ and $t = \sqrt[3]{e^{-1} - 1}$. Hence the domain of definition is

$$\left(\sqrt[3]{e^{-1}-1}, \infty\right)$$

Here is a graph for your amusement.