All multiple choice due to COVID issues. Each actual exam contained one part of parts (A), (B) ...

1.

(A) Which equation below is satisfied by the solution to $y' = e^{x+y}$ with initial condition $y(0) = 0$?

(a) $2e^{-2y} + e^x = 3$ (b) $e^x + e^{-y} = 2$ (c) $-e^y + 3e^{-x} = 2$ (d) $e^{-2y} + 2e^x = 3$

(e) $e^y - e^x = 0$

(B) Which equation below is satisfied by the solution to $y' = e^{x-y}$ with initial condition $y(0) = 0$?

(a) $2e^{-2y} + e^x = 3$ (b) $e^y - e^x = 0$ (c) $-e^y + 3e^{-x} = 2$ (d) $e^{-2y} + 2e^x = 3$

(e) $e^x + e^{-y} = 2$

(C) Which equation below is satisfied by the solution to $y' = e^{x+2y}$ with initial condition $y(0) = 0$?

(a) $2e^{-2y} + e^x = 3$ (b) $e^{-2y} + 2e^x = 3$ (c) $-e^y + 3e^{-x} = 2$ (d) $e^y - e^x = 0$

(e) $e^x + e^{-y} = 2$

2.

(A) Which function below is an integrating factor for the linear equation $xy' + (x + 1)y = e^x$?

(a) $x^{-1}e^x$ (b) $e^{x+x^2/2}$ (c) xe^{-x} (d) xe^x (e) $x^{-1}e^{-x}$

(B) Which function below is an integrating factor for the linear equation $xy' + (x - 1)y = e^x$?

(a) xe^x (b) $e^{x+x^2/2}$ (c) xe^{-x} (d) $x^{-1}e^x$ (e) $x^{-1}e^{-x}$

(C) Which function below is an integrating factor for the linear equation $xy' - (x + 1)y = e^x$?

(a) xe^x (b) $e^{x+x^2/2}$ (c) xe^{-x} (d) $x^{-1}e^{-x}$ (e) $x^{-1}e^x$

(D) Which function below is an integrating factor for the linear equation $xy' - (x-1)y = e^x$?

(a) xe^x (b) $e^{x+x^2/2}$ (c) $x^{-1}e^{-x}$ (d) xe^{-x} (e) $x^{-1}e^x$

3.

(A) The equation $(e^x + 4xy)dx + \big(2x^2 + \cos(y)\big)dy = 0$ is exact. Which formula below is an implicit solution which goes through the point $\left(-1, \frac{\pi}{2}\right)$?

(a) There is no solution to the equation going through the given point.

(b) $2x^2y + e^x + \sin(y) = \pi + 1 + e^{-1}$

(c) $2x^2y + e^{x+y} + \sin(x - y) = \pi - \cos(1) + e^{\pi/2-1}$

(d) $2x^2y + e^{x-y} + \sin(x + y) = \pi + \cos(1) + e^{\pi/2+1}$

(e) $2x^2y + e^y + \sin(x) = \pi - \sin(1) + e^{\pi/2}$

(B) The equation $(\cos(x) + 4xy)dx + \big(2x^2 + e^y\big)dy = 0$ is exact. Which formula below is an implicit solution which goes through the point $\left(-1, \frac{\pi}{2}\right)$?

(a) There is no solution to the equation going through the given point.

(b) $2x^2y + e^y + \sin(x) = \pi - \sin(1) + e^{\pi/2}$

(c) $2x^2y + e^{x+y} + \sin(x - y) = \pi - \cos(1) + e^{\pi/2-1}$

(d) $2x^2y + e^{x-y} + \sin(x + y) = \pi + \cos(1) + e^{\pi/2+1}$

(e) $2x^2y + e^x + \sin(y) = \pi + 1 + e^{-1}$

4.

(A) Classify all the equilibria of $y' = (3y^2 - 2y - 1)(\cos(y) + 2)$.

(a) No equilibrium is stable and none is unstable.

(b) All equilibria are unstable.

(c) $y = -\dfrac{1}{3}$ is stable; $y = 1$ is unstable.

(d) $y = -\dfrac{1}{3}$ is unstable; $y = 1$ is stable.

(e) All equilibria are stable.

(B) Classify all the equilibria of $y' = (3y^2 - 2y - 1)(\cos(y) - 2)$.

(a) No equilibrium is stable and none is unstable.

(b) All equilibria are unstable.

(c) $y = -\frac{1}{3}$ is unstable; $y = 1$ is stable.

(d) $y = -\frac{1}{3}$ is stable; $y = 1$ is unstable.

(e) All equilibria are stable.

5.

(A) What are the eigenvalues of the matrix $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 0 \\ -4 & 5 & 3 \end{bmatrix}$?

(a) $-3, 1, 3$ (b) $-2, 2, 4$ (c) $-1, 3, 5$ (d) $1, 3, 3$ (e) $0, 4, 6$

(B) What are the eigenvalues of the matrix $\begin{bmatrix} 2 & 4 & -2 \\ 0 & 4 & 0 \\ -4 & 5 & 4 \end{bmatrix}$?

(a) $-3, 1, 3$ (b) $-2, 2, 4$ (c) $0, 4, 6$ (d) $1, 3, 3$ (e) $-1, 3, 5$

(C) What are the eigenvalues of the matrix $\begin{bmatrix} 0 & 4 & -2 \\ 0 & 2 & 0 \\ -4 & 5 & 2 \end{bmatrix}$?

(a) $-3, 1, 3$ (b) $0, 4, 6$ (c) $-2, 2, 4$ (d) $1, 3, 3$ (e) $-1, 3, 5$

6.

(A) Which of the following is a complex eigenvector of the matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$ (b) $\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1-3i \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ (e) $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$

$\lambda^2 - 6\lambda + 10$

$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$

$\lambda = 3+i$ $A - (3+i)I = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$

$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ -1-i & 2 & 0 \end{array}\right] \xrightarrow{R_2 + (1+i)R_1} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & \underbrace{2+(1+i)(-1+i)}_{-2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_1, $x_2 = s$

$x_1 = (1-i)s$
$x_2 = s$

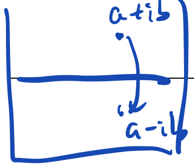
$s\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ conjugation

$\lambda = 3-i$ $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$\lambda = 3+i$ $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

complex conj.

$\lambda = 3-i$ $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$



$a+ib \to a-ib$

$2+3i \to 2-3i$

7.

(A) The vector $\vec{v}$ is an eigenvector with eigenvalue 2 for a square matrix A. Which of the following is equal to $(2A^3 - 3A^2)\vec{v}$?

(a) $28\vec{v}$ (b) $-28\vec{v}$ (c) $\vec{0}$ (d) $4\vec{v}$ (e) $-4\vec{v}$

(B) The vector $\vec{v}$ is an eigenvector with eigenvalue -2 for a square matrix A. Which of the following is equal to $(2A^3 - 3A^2)\vec{v}$?

(a) $28\vec{v}$ (b) $4\vec{v}$ (c) $\vec{0}$ (d) $-28\vec{v}$ (e) $-4\vec{v}$

(C) The vector $\vec{v}$ is an eigenvector with eigenvalue 2 for a square matrix A. Which of the following is equal to $(2A^3 + 3A^2)\vec{v}$?

(a) $-28\vec{v}$ (b) $4\vec{v}$ (c) $\vec{0}$ (d) $28\vec{v}$ (e) $-4\vec{v}$

(D) The vector $\vec{v}$ is an eigenvector with eigenvalue -2 for a square matrix A. Which of the following is equal to $(2A^3 + 3A^2)\vec{v}$?

(a) $-28\vec{v}$ (b) $4\vec{v}$ (c) $\vec{0}$ (d) $-4\vec{v}$ (e) $28\vec{v}$

8.

(A) Let A be the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, which has eigenvalues 2 and 5. Let m be the dimension of the 2-eigenspace of A and n be the dimension of the 5-eigenspace of A. Which of the following is true?

(a) $m = 2, n = 1$ (b) $m = 1, n = 2$ (c) $m = 2, n = 2$ (d) $m = 2, n = 0$ (e) $m = 1, n = 1$

(B) Let A be the matrix $\begin{bmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix}$, which has eigenvalues 5 and 8. Let m be the dimension of the 8-eigenspace of A and n be the dimension of the 5-eigenspace of A. Which of the following is true?

(a) $m = 1, n = 2$ (b) $m = 2, n = 1$ (c) $m = 2, n = 2$ (d) $m = 2, n = 0$ (e) $m = 1, n = 1$

9.

(A) Let A be a 5×5 matrix whose only eigenvalues are 1 and 2. Four of the statements below are false and one is sometimes true and sometimes false. Which one COULD be true?

(a) The algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 1 and 4, , and the geometric multiplicities are 0 and 3, respectively.

(b) The algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 2 and 3, and their geometric multiplicities are 3 and 2, respectively.

(c) The algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 2 and 3, and the geometric multiplicities are 2 and 2, respectively.

(d) The algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 2 , and 4, and the geometric multiplicities are 2 and 4, respectively.

(e) The algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 2 and 2, and the geometric multiplicities are 2 and 3, respectively.

10.

(A) Which among the following statements is FALSE?

(a) Any 1×1 matrix is diagonalizable. true

(b) If matrices A and B are similar, then $\det A = \det B$. true

$B = PAP^{-1}$

$\det B = \det P \det A (\det P)^{-1}$

(c) If the characteristic polynomial of an $n \times n$ matrix A has n distinct roots, then A is diagonalizable. true

(d) Any lower-triangular matrix is diagonalizable. false

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(e) The only matrix similar to the identity matrix I_n is I_n itself. true

$PI_nP^{-1} = PP^{-1} = I_n$

$A\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 \quad x_2 = s$

$m_{alg} = 2$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\lambda = 0$ eigenval

$\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 = 0$

11.

(A) Given the vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

the Gram-Schmidt algorithm for the first two of them yields

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4/3 \\ -2/3 \\ -2/3 \end{bmatrix}.$$

Determine the third vector $\vec{v}_3$.

(a) Since the vectors are linearly dependent, $\vec{v}_3 = 0$

(b) $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(c) $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(d) $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(e) $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

(B) Given the vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

the Gram-Schmidt algorithm for the first two of them yields

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2/3 \\ -4/3 \\ 2/3 \end{bmatrix}.$$

Determine the third vector $\vec{v}_3$.

(a) Since the vectors are linearly dependent, $\vec{v}_3 = 0$

(b) $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(c) $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(d) $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(e) $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

12.

(A) Given a subspace $W \subseteq \mathbb{R}^4$ with some orthogonal basis $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and a vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \in \mathbb{R}^4$, compute $\text{proj}_W \vec{v}$.

(a) $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(B) Given a subspace $W \subseteq \mathbb{R}^4$ with some orthogonal basis $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, and a vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \in \mathbb{R}^4$, compute $\text{proj}_W \vec{v}$.

(a) $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

13.

(A) Given a matrix $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and its QR-decomposition

$$Q = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{6}/6 & 0 \\ 0 & \sqrt{6}/3 & 0 \\ \sqrt{2}/2 & -\sqrt{6}/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \text{a} & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \text{b} \\ 0 & 0 & 1 \end{bmatrix},$$

determine the missing entries a and b.

(a) $a = (3\sqrt{2} - \sqrt{6})/6$, $b = \sqrt{6}/3$

(b) $a = \sqrt{2}/2$, $b = -\sqrt{6}/2$

(c) $a = 0$, $b = \sqrt{6}$

(d) $a = -\sqrt{6}/6$, $b = 0$

(e) $a = -\sqrt{2}/2$, $b = \sqrt{6}/2$

(B) Given a matrix $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and its QR-decomposition

$$Q = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{6}/6 & 0 \\ 0 & \sqrt{6}/3 & 0 \\ \sqrt{2}/2 & -\sqrt{6}/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \text{a} & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \text{b} \\ 0 & 0 & 1 \end{bmatrix},$$

determine the missing entries a and b.

(a) $a = (3\sqrt{2} - \sqrt{6})/6$, $b = \sqrt{6}/3$

(b) $a = -\sqrt{2}/2$, $b = \sqrt{6}/2$

(c) $a = 0$, $b = \sqrt{6}$

(d) $a = -\sqrt{6}/6$, $b = 0$

(e) $a = \sqrt{2}/2$, $b = -\sqrt{6}/2$

14.

(A) Given a subspace W of $\mathbb{R}^4$

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right),$$

determine a basis of $W^\perp$.

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

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(c) $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

(B) Given a subspace W of $\mathbb{R}^4$

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right),$$

determine a basis of $W^{\perp}$.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

15.

(A) Find a least squares solution to $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

(a) $\overline{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\overline{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(c) $\overline{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(d) No least squares solution exists as this is an inconsistent system.

(e) $\overline{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(B) Find a least squares solution to $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

(a) $\overline{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\overline{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(c) $\overline{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(d) No least squares solution exists as this is an inconsistent system.

(e) $\overline{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

16.

(A) We would like to find an approximating line, $y = a + bx$, for the data points

$$\{(-1,0)\ ,\ (0,0)\ ,\ (2,1)\}?$$

Which of the following are the normal equations that will solve for the line of best fit?

(a) $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \overline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} \overline{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(B) We would like to find an approximating line, $y = a + bx$, for the data points

$$\{(1,0)\ ,\ (0,0)\ ,\ (-2,1)\}?$$

Which of the following are the normal equations that will solve for the line of best fit?

(a) $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} \overline{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \overline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \overline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

17.

(A) Find the standard matrix for the linear transformation of $\text{proj}_W : \mathbb{R}^3 \to \mathbb{R}^3$ for subspace W with basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) $\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(b) There is not a standard matrix since proj_W is not linear.

(c) $\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(e) $\frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(B) Find the standard matrix for the linear transformation of $\text{proj}_W : \mathbb{R}^3 \to \mathbb{R}^3$ for subspace W with basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

(a) $\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(b) There is not a standard matrix since proj_W is not linear.

(c) $\frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(e) $\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solutions

1. 1 $y' = e^{x+y} = e^x e^y$ so $e^{-y}dy = e^x dx$ or $-e^{-y} = e^x + C$; $e^x + e^{-y} = A$. At $(0,0)$, $A = 2$.

1. 2 $y' = e^{x-y} = e^x e^{-y}$ so $e^y dy = e^x dx$ or $e^y = e^x + C$; $e^y - e^x = C$. At $(0,0)$, $C = 0$.

1. 3 $y' = e^{x+2y} = e^x e^{2y}$ so $e^{-2y}dy = e^x dx$ or $-\frac{1}{2}e^{-2y} = e^x + C$; $e^{-2y} + 2e^x = A$. At $(0,0)$, $A = 3$.

2. 1 Standard form: $y' + \frac{x+1}{x}y = \frac{e^x}{x}$. $\frac{\mu'}{\mu} = \frac{x+1}{x}$ so $\ln|\mu| = \int 1 + \frac{1}{x}\, dx = x + \ln|x| + C$ so $\mu = \pm e^{x+\ln|x|} = |x|e^x$ so we can use $\mu = x\, e^x$.

2. 2 Standard form: $y' + \frac{x-1}{x} y = \frac{e^x}{x}$. $\frac{\mu'}{\mu} = \frac{x-1}{x}$ so $\ln|\mu| = \int 1 - \frac{1}{x}\, dx = x - \ln|x| + C$ so $\mu = \pm e^{x - \ln|x|} = \frac{e^x}{|x|}$ so we can use $\mu = x^{-1}e^x$.

2. 3 Standard form: $y' - \frac{x+1}{x} y = \frac{e^x}{x}$. $\frac{\mu'}{\mu} = -\frac{x+1}{x}$ so $\ln|\mu| = \int -1 - \frac{1}{x}\, dx = -x - \ln|x| + C$ so $\mu = \pm e^{-x-\ln|x|} = \frac{e^{-x}}{|x|}$ so we can use $\mu = x^{-1}e^{-x}$.

2. 4 Standard form: $y' - \frac{x-1}{x} y = \frac{e^x}{x}$. $\frac{\mu'}{\mu} = -\frac{x-1}{x}$ so $\ln|\mu| = \int -1 + \frac{1}{x}\, dx = -x + \ln|x| + C$ so $\mu = \pm e^{-x+\ln|x|} = e^{-x}|x|$ so we can use $\mu = xe^{-x}$.

3. 1 $M(x,y) = e^x + 4xy$; $N(x,y) = 2x^2 + \cos(y)$.
$\frac{\partial M}{\partial y} = 4x$; $\frac{\partial N}{\partial x} = 4x$.
$m(x,y) = 2x^2y + e^x + h(y)$ so $\frac{\partial m}{\partial y} = 2x^2 + 0 + h'(y)$.
$m_y(x,y) = 2x^2 + 0 + h'(y) = 2x^2 + \cos(y)$.
$m(x,y) = 2x^2y + e^x + \sin(y) + K$.

$$2x^2y + e^x + \sin(y) = C$$

$C = 2 \cdot (-1)^2 \cdot \left(\frac{\pi}{2}\right) + e^{-1} + \sin\left(\frac{\pi}{2}\right) = \pi + 1 + e^{-1}$ so

$$2x^2y + e^x + \sin(y) = \pi + 1 + e^{-1}$$

3. 2

$M(x,y) = \cos(x) + 4xy$, $N(x,y) = 2x^2 + e^y$ so $\dfrac{\partial M}{\partial y} = 4x$; $\dfrac{\partial N}{\partial x} = 4x$.

$m(x,y) = 2x^2y + \sin(x) + h(y)$ so $\dfrac{\partial m}{\partial y} = 2x^2 + 0 + h'(y)$.

$m_y(x,y) = 2x^2 + 0 + h'(y) = 2x^2 + e^y$.

$m(x,y) = 2x^2y + e^y + \sin(x) + K$.

$$2x^2y + e^y + \sin(x) = C$$

$C = 2 \cdot (-1)^2 \cdot \left(\frac{\pi}{2}\right) + e^{\frac{\pi}{2}} + \sin(1) = \pi + \sin(-1) + e^{\frac{\pi}{2}}$ so

$$2x^2y + e^y + \sin(x) = \pi + \sin(-1) + e^{\pi/2}$$

4. 1

Let $F(y) = (3y^2-2y-1)(\cos(y)+2)$. Since $\cos(y)+2 > 0$ the equilibria are the solutions to $3y^2-2y-1 = (3y+1)(y-1) = 0$ so $y = 1$ and $y = -\frac{1}{3}$ are the equilibria. $F(0) = -1(3) < 0$; $F(2) = 7(\cos(2)+2) > 0$ and $F(-1) = 4(\cos(-1)+2) > 0$. Hence $y = -\frac{1}{3}$ is a stable equilibrium and $y = 1$ is unstable.

4. 2

Let $F(y) = (3y^2-2y-1)(\cos(y)-2)$. Since $\cos(y)-2 < 0$ the equilibria are the solutions to $3y^2-2y-1 = (3y+1)(y-1) = 0$ so $y = 1$ and $y = -\frac{1}{3}$ are the equilibria. $F(0) = (-1)(-1) > 0$; $F(2) = 7(\cos(2)-2) < 0$ and $F(-1) = 4(\cos(-1)-2) < 0$. Hence $y = -\frac{1}{3}$ is an unstable equilibrium and $y = 1$ is stable.

5. 1

The characteristic polynomial of the matrix is

$$\begin{vmatrix} 1-\lambda & 4 & -2 \\ 0 & 3-\lambda & 0 \\ -4 & 5 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left((1-\lambda)(3-\lambda)-(-2)(-4)\right)$$

$$= (3-\lambda)(\lambda^2-4\lambda-5) = (3-\lambda)(\lambda-5)(\lambda+1)$$

which has roots $\lambda = -1, 3, 5$. The eigenvalues are -1, 3 and 5.

5. 2

The characteristic polynomial of the matrix is

$$\begin{vmatrix} 2-\lambda & 4 & -2 \\ 0 & 4-\lambda & 0 \\ -4 & 5 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -4 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2-6\lambda+0) = (4-\lambda)\lambda(\lambda-6)$$

which has roots $\lambda = 0, 4, 6$. The eigenvalues are 0, 4 and 6.

5. 3

The characteristic polynomial of the matrix is

$$\begin{vmatrix} -\lambda & 4 & -2 \\ 0 & 2-\lambda & 0 \\ -4 & 5 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2-2\lambda-8) = (2-\lambda)(\lambda-4)(\lambda+2)$$

which has roots $\lambda = -2, 2, 4$. The eigenvalues are -2, 2 and 4.

6. The characteristic polynomial of the 2×2-matrix A is given by $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 10 = (\lambda - 3)^2 + 1$ with roots $3 \pm i$. We consider the eigenvalue $3 - i$. Then $A - \lambda \mathrm{I}_2 = \begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}$.
A corresponding eigenvector $[x_1\ x_2]^T$ satisfies $-1x_1 + (1+i)x_2 = 0$ so could be taken to be $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.
(Working with the conjugate eigenvalue $3 + i$ would give the conjugate eigenvector, but you should still recognize the correct answer from the fact that complex conjugates of eigenvectors of real matrices are also eigenvectors, for the complex conjugate eigenvalue.)
Alternatively, check that $\begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = (3-i) \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ to see that $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $3 - i$.

7. 1 Given any polynomial $p(x)$, $p(A)\vec{v} = p(2)\vec{v}$. In the problem, $p(x) = 2x^3 - 3x^2$ and $p(2) = 16 - 12 = 4$.

7. 2 Given any polynomial $p(x)$, $p(A)\vec{v} = p(-2)\vec{v}$. In the problem, $p(x) = 2x^3 - 3x^2$ and $p(-2) = -16 - 12 = -28$.

7. 3 Given any polynomial $p(x)$, $p(A)\vec{v} = p(2)\vec{v}$. In the problem, $p(x) = 2x^3 + 3x^2$ and $p(2) = 16 + 12 = 28$.

7. 4 Given any polynomial $p(x)$, $p(A)\vec{v} = p(-2)\vec{v}$. In the problem, $p(x) = 2x^3 + 3x^2$ and $p(-2) = -16 + 12 = -4$.

8. 1

$$A - 2I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A - 5I_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

The case $\lambda = 2$ looks easiest to reduce: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so the 2-eigenspace has dimension 2. Hence the 5-eigenspace has dimension at least 1 and at most $3 - 2$ so it has dimension 1.

For those of you who did row reduce $\lambda = 5$, we include the computation:

$$A - \lambda I_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

8. 2

$$A - 5I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A - 8I_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

The case $\lambda = 5$ looks easiest to reduce: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so the 5-eigenspace has dimension 2. Hence the 5-eigenspace has dimension at least 1 and at most $3 - 2$ so it has dimension 1.

9. The sum of the algebraic multiplicities of the complex eigenvalues is the size of the matrix (5) so algebraic multiplicities of 2 and 2, or 2 and 4, are impossible. Geometric multiplicities of eigenvalues are between 1 and the algebraic multiplicity (inclusive), so a geometric multiplicity of 0 is impossible, and a geometric multiplicity greater than the corresponding algebraic multiplicity is impossible. The case in which the algebraic multiplicities of the eigenvalues $\lambda = 1$ and $\lambda = 2$ are 2 and 3, and the geometric multiplicities are 2 and 2, respectively, meets all these requirements and is possible.

10. If A and B are similar, $A = PBP^{-1}$ and so they have the same determinant. If the characteristic polynomial of an $n \times n$ matrix A has n distinct roots, then the fact that A is diagonalizable is a theorem. If $A = PI_nP^{-1}$ then $A = I$. Any 1×1 matrix is obviously diagonalizable. This only leaves lower triangular matrices so there must be some which are not diagonalizable: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has characteristic polynomial x^2 so if it is diagonalizable, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix}$ so $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable.

11. 1 According to the Gram-Schmidt algorithm, the third vector is given by

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{v}_1 \bullet \vec{x}_3}{\vec{v}_1 \bullet \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \bullet \vec{x}_3}{\vec{v}_2 \bullet \vec{v}_2}\vec{v}_2.$$

Plugging in the known vectors, we obtain

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \frac{4/3}{24/9}\begin{bmatrix} -4/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

11. 2

Just for completeness check $\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{x}_1 \bullet \vec{x}_2}{\vec{x}_1 \bullet \vec{x}_1}\vec{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - (-1/3)\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -4/3 \\ 2/3 \end{bmatrix}$.

Then

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{v}_1 \bullet \vec{x}_3}{\vec{v}_1 \bullet \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \bullet \vec{x}_3}{\vec{v}_2 \bullet \vec{v}_2}\vec{v}_2.$$

Plugging in the known vectors, we obtain

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \frac{4/3}{24/9}\begin{bmatrix} -2/3 \\ -4/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2/3 \\ -4/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

12. 1

According to the theory, the projection of $\vec{v}$ onto W is given by

$$\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \text{proj}_{\vec{u}_2} \vec{v}.$$

Therefore,

$$\text{proj}_W \vec{v} = \frac{1-1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{-1-1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$$

12. 2

According to the theory, the projection of $\vec{v}$ onto W is given by

$$\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \text{proj}_{\vec{u}_2} \vec{v}.$$

Therefore,

$$\text{proj}_W \vec{v} = \frac{1+1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{-1+1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

13. 1

From the formula $A = QR$, we solve for R as $R = Q^T A$. Therefore, the $(1, 2)$ entry of R (which is a) is given by the dot product of the first column of Q with the second column of A:

$$a = (-\sqrt{2}/2) \cdot (-1) = \sqrt{2}/2.$$

Similarly, the $(2, 3)$ entry of R (which is b) is given by the dot product of the second column of Q with the third column of A:

$$b = \sqrt{6}/3 \cdot (-1) + (-\sqrt{6}/6) \cdot 1 = -\sqrt{6}/2.$$

13. 2

From the formula $A = QR$, we solve for R as $R = Q^T A$. Therefore, the $(1, 2)$ entry of R (which is a) is given by the dot product of the first column of Q with the second column of A:

$$a = \sqrt{2}/2 \cdot (-1) = -\sqrt{2}/2.$$

Similarly, the $(2, 3)$ entry of R (which is b) is given by the dot product of the second column of Q with the third column of A:

$$b = (-\sqrt{6}/6) \cdot (-1) + \sqrt{6}/3 \cdot 1 = \sqrt{6}/2.$$

14. 1

If $\vec{x} \in W^\perp$, then $\vec{x}$ is orthogonal to the basis of W, and this yields a system of linear equations for the components of $\vec{x}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 0;$$

it becomes

$$\begin{cases} x_1 + 2x_3 - x_4 = 0 \\ x_2 + x_3 - x_4 = 0. \end{cases}$$

This system is already in its RREF, so we choose $t := x_3$ and $s := x_4$, and the leading variables are then given by

$$\begin{cases} x_1 = -2t + s \\ x_2 = -t + s \end{cases}$$

Therefore, the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t + s \\ -t + s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis of $W^\perp$ can be chosen as $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

14. 2

If $\vec{x} \in W^{\perp}$, then $\vec{x}$ is orthogonal to the basis of W, and this yields a system of linear equations for the components of $\vec{x}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0;$$

it becomes

$$\begin{cases} x_1 + 2x_3 - x_4 = 0 \\ x_2 - x_3 = 0. \end{cases}$$

This system is already in its RREF, so we choose $t := x_3$ and $s := x_4$, and the leading variables are then given by

$$\begin{cases} x_1 = -2t + s \\ x_2 = t. \end{cases}$$

Therefore, the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t + s \\ t \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis of $W^{\perp}$ can be chosen as $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

15. 1

The least squares solution can be found by solving the system

$$A^T A\overline{x} = A^T\vec{b}.$$

Since A has linearly independent columns, there is a unique solution to this system. We compute $A^T A$ and $A^T\vec{b}$:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

We can solve the system $(A^T A)\overline{x} = A^T\vec{b}$ directly or find

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Using this we can solve for $\overline{x}$,

$$\overline{x} = (A^T A)^{-1}(A^T\vec{b}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

15. 2

The least squares solution can be found by solving the system
$$A^TA\overline{x} = A^T\vec{b}.$$
Since A has linearly independent columns, there is a unique solution to this system. We compute A^TA and $A^T\vec{b}$:
$$A^TA = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},$$
$$A^T\vec{b} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\ 5\\ 1\end{bmatrix} = \begin{bmatrix}3\\ 6\end{bmatrix}.$$
We can solve the system $(A^TA)\overline{x} = A^T\vec{b}$ directly or find
$$(A^TA)^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}.$$
Using this we can solve for $\overline{x}$,
$$\overline{x} = (A^TA)^{-1}(A^T\vec{b}) = \frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3\\ 6\end{bmatrix} = \begin{bmatrix}0\\ 3\end{bmatrix}.$$

16. 1

We are using least squares approximation to find a line $y = a + bx$ that minimizes the error vector for these data points. Evaluating the line's equation at the data points, we get the following three equations:
$$\begin{aligned} a - b &= 0\\ a &= 0\\ a + 2b &= 1 \end{aligned}$$

To find a and b for the line of best fit, we find a least squares solution to the inconsistent system
$$\begin{bmatrix}1 & -1\\ 1 & 0\\ 1 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$
Let
$$A = \begin{bmatrix}1 & -1\\ 1 & 0\\ 1 & 2\end{bmatrix}, \vec{b} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$
Then the least squares solution to $A\vec{x} = \vec{b}$ is the solution $\overline{x}$ to system $A^TA\overline{x} = A^T\vec{b}$. Computing $A^TA = \begin{bmatrix}1 & 1 & 1\\ -1 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}3 & 1\\ 1 & 5\end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix}1 & 1 & 1\\ -1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}$, we find
$$\begin{bmatrix}3 & 1\\ 1 & 5\end{bmatrix}\overline{x} = \begin{bmatrix}1\\ 2\end{bmatrix}.$$

16. 2

We are using least squares approximation to find a line $y = a + bx$ that minimizes the error vector for these data points. Evaluating the line's equation at the data points, we get the following three equations:

$$\begin{aligned} a + b &= 0 \\ a &= 0 \\ a - 2b &= 1 \end{aligned}$$

To find a and b for the line of best fit, we find a least squares solution to the inconsistent system

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the least squares solution to $A\vec{x} = \vec{b}$ is the solution $\overline{x}$ to system $A^T A\overline{x} = A^T\vec{b}$. Computing $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, we find

$$\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} \overline{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

17. 1

From the basis of W, we construct a matrix A,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $\text{proj}_W(\vec{v}) = A(A^T A)^{-1} A^T\vec{v}$. In particular, the standard matrix of proj_W is $A(A^T A)^{-1} A^T$. We compute $A^T A$ and its inverse first:

$$\begin{aligned} A^T A &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \\ (A^T A)^{-1} &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \end{aligned}$$

Now we find

$$A(A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

17. 2

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\left(A^T A\right)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Then

$$\text{proj}_W = A(A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$