BF theory on cobordisms endowed with cellular decomposition

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Joint work with Alberto S. Cattaneo and Nikolai Reshetikhin

1. BV-BFV formalism for gauge theories on manifolds with boundary: an outline.

Plan

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2. Cellular abelian BF theory.

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3. Cellular non-abelian BF theory

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Note: $\{S, S\}_\omega = 0$.

BV-BFV formalism for gauge theories on manifolds with boundary

Reference: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.

For M with boundary:

$$\begin{tikzcd}
M \arrow[r] & (\mathcal{F}, \arrow[d, "\pi"] & \omega, & Q, \arrow[d, "\pi_*"] & S) & \text{– space of fields} \\
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Subscripts = “ghost numbers”.

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Relations: $Q_\partial^2 = 0,\ \iota_{Q_\partial}\omega_\partial = \delta S_\partial;\qquad Q^2 = 0,\ \boxed{\iota_Q\omega = \delta S + \pi^*\alpha_\partial}.$

$\Rightarrow$CME: $\frac{1}{2}\iota_Q\iota_Q\omega = \pi^* S_\partial$

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$$M_I \cup_\Sigma M_{II} \to \mathcal{F}_{M_I} \times_{\Phi_\Sigma} \mathcal{F}_{M_{II}}$$

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This picture extends to higher-codimension strata!

Quantum BV-BFV formalism.

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Reminder: In Darboux coordinates (x^i, ξ_i) on $\mathcal{F}_{\mathrm{res}}$,

$$\Delta_{\mathrm{res}} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}$$

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P_* — BV pushforward (fiber BV integral) for

$\mathcal{F}^{M_I}_{\mathrm{res}} \times \mathcal{F}^{M_{II}}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}^{M_I\cup_\Sigma M_{II}}_{\mathrm{res}}$

Quantization

Choose $p: \Phi_\partial \to \mathcal{B}_\partial$ Lagrangian fibration, $\alpha_\partial|_{p^{-1}(b)} = 0$.

$$\boxed{\mathcal{H}_\partial = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}_\partial)}, \ \Omega_\partial = \widehat{S_\partial} \quad \in \mathrm{End}(\mathcal{H}_\partial)_1.$$

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Partition function:

$$Z_M(b) = \int_{\mathcal{L}\subset\mathcal{F}_b} e^{\frac{i}{\hbar}S}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}_\partial)$$

$\mathcal{L} \subset \mathcal{F}_b$ gauge-fixing Lagrangian.
Problem: Z_M may be ill-defined due to zero-modes.

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Solution: Split $\mathcal{F}_b = \mathcal{F}_{\mathrm{res}} \times \widetilde{\mathcal{F}} \ni (\phi_{\mathrm{res}}, \widetilde{\phi})$. Partition function:

$$Z_M(b, \phi_{\mathrm{res}}) = \int_{\mathcal{L} \subset \widetilde{\mathcal{F}}} e^{\frac{i}{\hbar} S(b, \phi_{\mathrm{res}}, \widetilde{\phi})}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}_\partial) \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{\mathrm{res}})$$

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$$\mathcal{F}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}'_{\mathrm{res}} \qquad \Rightarrow \qquad Z'_M = P_* Z_M$$

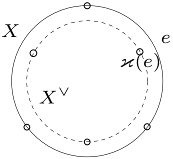
Cellular abelian BF theory – premise

Reference: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Cellular BV-BFV-BF theory*, in preparation.

- M - compact oriented PL n-manifold.
- X - cellular decomposition of M.
- Fields: differential forms → cellular cochains. $\mathcal{F} = C^\bullet(X)[1] \oplus C^\bullet(X^\vee)[n-2]$.

Dual cellular decomposition

- M closed



$\varkappa$: k-cells of $X \leftrightarrow (n-k)$-cells of $X^\vee$

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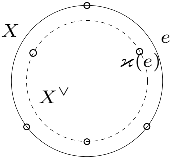

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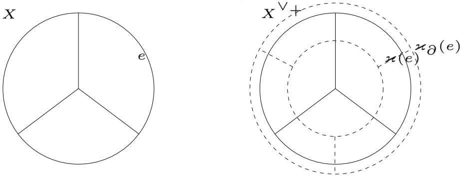

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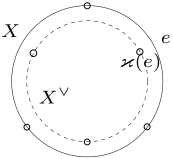

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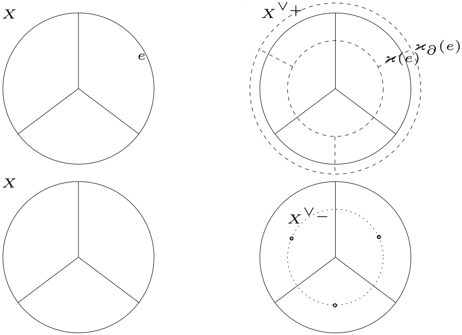

Dual cellular decomposition

Intersection numbers

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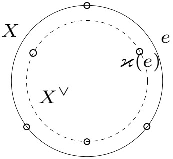

$$C_k(X) \otimes C_{n-k}(X^\vee) \to \mathbb{Z}$$

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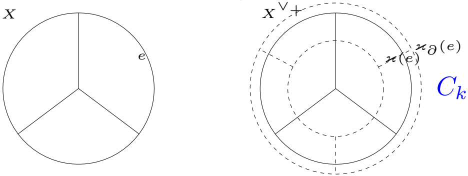

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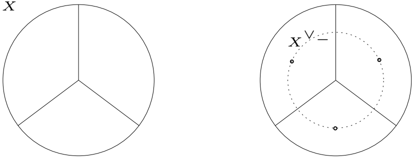

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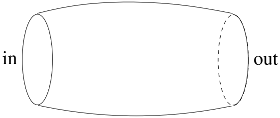
Dual cellular decomposition: cobordisms

M a cobordism



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X – cellular decomposition of M

Dual cellular decomposition: cobordisms

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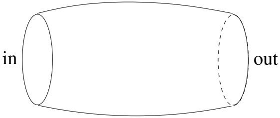

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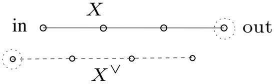

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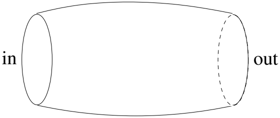

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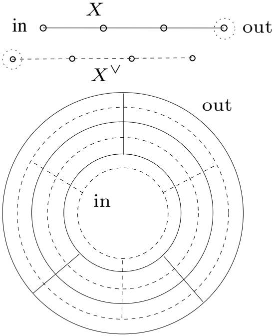

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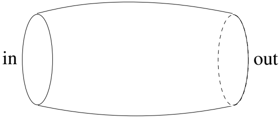

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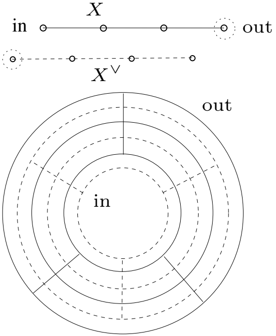

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Cellular abelian BF: fields and bulk 2-form

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- In components:
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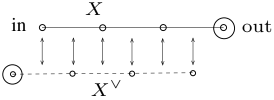

$\ker \omega$ spanned by A_{out}, B_{in}.

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$$\sum_{e\subset X-X_{\mathrm{out}}} \sum_{e' \underset{\mathrm{codim}=0}{\subset} \partial e} \pm \langle B_{\varkappa(e)}, E(e>e')\circ A_{e'}\rangle_{\mathbb{R}^m} + \langle B, A\rangle_{\mathrm{in}}$$

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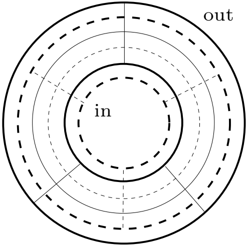

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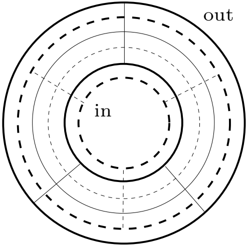

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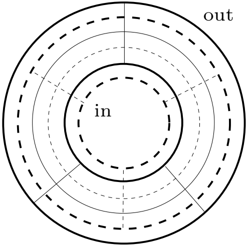

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Q_∂ – as in bulk, but for cochains of the boundary, $S_\partial = \langle B, d_E A\rangle_\partial$.

Lemma

Bulk and boundary data $(\mathcal{F}, \omega, Q, S, \pi)$, $(\Phi_\partial, \omega_\partial = \delta\alpha_\partial, Q_\partial, S_\partial)$ introduced above satisfies the properties of a classical BV-BFV theory

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E.g. **structure equation** $\boxed{\iota_Q\omega=\delta S+\pi^*\alpha_\partial}$ – corollary of **cellular Stokes' theorem** $\langle db,a\rangle\pm\langle b,da\rangle=\langle b,a\rangle_{\mathrm{out}}-\langle b,a\rangle_{\mathrm{in}}$ with b,a test cochains.

Quantization for M closed

Set $\mu^{\frac{1}{2}} = \prod_{e\subset X} \prod_{a=1}^{m} |\mathcal{D}A_e^a|^{\frac{1}{2}}\, |\mathcal{D}B_{\varkappa(e)a}|^{\frac{1}{2}} \quad \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$

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The space of choices of (i,p,K) is contractible.

Partition function for M closed, “renormalization” of integration measure

$Z = \tau(X,E) \cdot \zeta_{X,E}$ where

- $\tau(X,E) = \tau(M,E)$ – Reidemeister torsion, independent of X.
- $\zeta_{X,E} = (2\pi\hbar)^{\frac{1}{2}\dim \mathcal{L}^{\mathrm{even}}} \cdot \left(\frac{i}{\hbar}\right)^{\frac{1}{2}\dim \mathcal{L}^{\mathrm{odd}}}$ depends on X.

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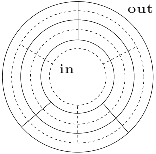
We can renormalize the integration measure $\mu^{\frac{1}{2}} \to \xi_{C^\bullet}\cdot\mu^{\frac{1}{2}} =: \mu_\hbar^{\frac{1}{2}}$, then

$Z \to \boxed{Z_{\mathrm{new}} = \tau(M,E)\cdot\xi_{H^\bullet}} \quad \in \mathbb{C}\otimes\mathrm{Det}H^\bullet(M,E)/\{\pm 1\}$

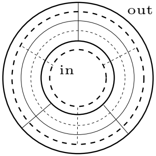
– indepenent of X, contains a $\mathrm{mod}16$ complex phase.

Quantization for M a cobordism



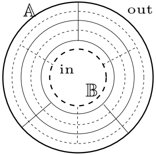
$$\mathcal{F} = C^\bullet(X)[1] \oplus C^\bullet(X^\vee)[n-2] \quad \supset \mathcal{F}_b = \left\{ (A,B) \;\middle|\; \begin{array}{l} A|_{\text{out}} = \mathbb{A} \\ B|_{\text{in}} = \mathbb{B} \end{array} \right\}$$

$$\pi \Big\downarrow$$



$$\Phi_\partial = C^\bullet(X_\partial)[1] \oplus C^\bullet(X_\partial^\vee)[n-2]$$

$$p \Big\downarrow \text{polarization}$$



$$\mathcal{B}_\partial = C^\bullet(X_{\text{out}})[1] \oplus C^\bullet(X_{\text{in}}^\vee)[n-2] \qquad \ni b = (\mathbb{A}, \mathbb{B})$$

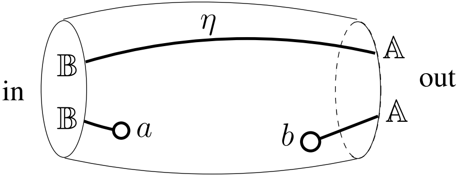
- $\mathcal{F}_b \simeq \mathcal{V} = C^\bullet(X, X_{\text{out}})[1] \oplus C^\bullet(X^\vee, X^\vee_{\text{in}})[n-2]$,
  Fields split as $\mathcal{F} \simeq \mathcal{B}_\partial \oplus \mathcal{V}$.
- Note: $\omega_b := \omega|_{\mathcal{F}_b}$ is non-degenerate.

Space of states on the boundary: $\mathcal{H}_\partial := \text{Dens}^{\frac{1}{2}}(\mathcal{B}_\partial)$,
Differential on states (BFV charge):
$\Omega_\partial := -i\hbar\, p_* Q_\partial = -i\hbar \left(\langle d_E \mathbb{A}, \frac{\partial}{\partial \mathbb{A}} \rangle + \langle d^\vee_{E^\vee} \mathbb{B}, \frac{\partial}{\partial \mathbb{B}} \rangle \right)$
Modified quantum master equation:
$\left(-i\hbar \Delta_{\mathcal{V}} + \frac{i}{\hbar}\Omega_\partial\right) \circ \left(e^{\frac{i}{\hbar} S} \cdot \mu_\hbar^{\frac{1}{2}} \right) = 0$
Residual fields: $\mathcal{F}_{\text{res}} := H^\bullet(M, M_{\text{out}})[1] \oplus H^\bullet(M, M_{\text{in}})[n-2] \quad \ni (a, b)$.
Gauge-fixing: choice $C^\bullet(X, X_{\text{out}}) \overset{(i,p,K)}{\rightsquigarrow} H^\bullet(M, M_{\text{out}})$ induces splitting $\mathcal{V} \simeq \mathcal{F}_{\text{res}} \oplus \widetilde{\mathcal{F}}$ and Lagrangian $\mathcal{L} \subset \widetilde{\mathcal{F}}$.
Partition function:
$Z(\mathbb{A}, \mathbb{B}; a, b) = \int_{\mathcal{L} \subset \widetilde{\mathcal{F}}} e^{\frac{i}{\hbar} S(\mathbb{A} + a + \widetilde{A}, \mathbb{B} + b + \widetilde{B})} \cdot \mu_\hbar^{\frac{1}{2}} \quad \in \text{Dens}^{\frac{1}{2}}(\mathcal{F}_{\text{res}}) \otimes \mathcal{H}_\partial$

Partition function for M a cobordism

$$Z = \xi_{H^\bullet(M,M_{\text{out}})} \cdot \mu^{\frac{1}{2}}_{\mathcal{B}_\partial,\hbar} \cdot \tau(M, M_{\text{out}}) \cdot$$

$$\cdot \exp \frac{i}{\hbar} \left(\langle b, \mathbb{A} \rangle_{\text{out}} + \langle \mathbb{B}, a \rangle_{\text{in}} - \sum_{e \subset X_{\text{in}}} \sum_{e' \subset X_{\text{out}}} \mathbb{B}_{\varkappa_{\text{in}}(e)} \eta(e, \varkappa(e'))\, \mathbb{A}_{e'} \right)$$

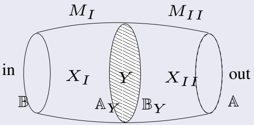


Where

- $\eta \in C^{n-1}(X \times X^\vee)$ is the matrix of K (the **parametrix** or **propagator**).
- $\mu^{\frac{1}{2}}_{\mathcal{B}_\partial,\hbar} = \prod_{e \subset X_{\text{out}}} |\mathcal{D}_\hbar \mathbb{A}_e|^{\frac{1}{2}} \cdot \prod_{e^\vee \subset X^\vee_{\text{in}}} |\mathcal{D}_\hbar \mathbb{B}_{e^\vee}|^{\frac{1}{2}}$ is the renormalized cellular $\frac{1}{2}$-density on $\mathcal{B}_\partial$.

Theorem: abelian cellular BF is a quantum BV-BFV theory

1. mQME: $\left(-i\hbar\Delta_{\mathrm{res}}+\frac{i}{\hbar}\Omega_\partial\right)\circ Z=0.$
2. Change of gauge-fixing induces a change $Z\to Z+\left(-i\hbar\Delta_{\mathrm{res}}+\frac{i}{\hbar}\Omega_\partial\right)\circ(\cdots).$
3. Gluing: for $X=X_I\cup_Y X_{II}$



we have $\boxed{Z_X=P_*(Z_{X_I}*_Y Z_{X_{II}})}$ where $Z_{X_I}*_Y Z_{X_{II}}=$

$$\int_{\mathbb{A}_Y,\mathbb{B}_Y} Z_{X_I}(\mathbb{B},\mathbb{A}_Y;a_I,b_I)\,\mathcal{D}^{\frac{1}{2}}\mathbb{A}_Y\,e^{-\frac{i}{\hbar}\langle\mathbb{B}_Y,\mathbb{A}_Y\rangle}\mathcal{D}^{\frac{1}{2}}\mathbb{B}_Y\,Z_{X_{II}}(\mathbb{B}_Y,\mathbb{A};a_{II},b_{II})$$

P_* – BV pushforward along $\mathcal{F}^I_{\mathrm{res}}\times\mathcal{F}^{II}_{\mathrm{res}}\to\mathcal{F}^{I\cup II}_{\mathrm{res}}$.

4. Z modulo $\left(-i\hbar\Delta_{\mathrm{res}}+\frac{i}{\hbar}\Omega_\partial\right)\circ(\cdots)$ is independent of X if X_{in}, X_{out} are fixed.

Remark: one can pass to **reduced space of states**
$H^\bullet_{\Omega_\partial}(\mathcal{H}_\partial)=\mathrm{Dens}^{\frac{1}{2}}(H^\bullet(M_{\mathrm{out}})[1]\oplus H^\bullet(M_{\mathrm{in}})[n-2]),$
then Z^{reduced} is completely independent of X (including boundary).

Non-abelian model

Fix $\mathfrak{g} = \mathrm{Lie}(G)$ a unimodular Lie algebra.

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Continuum non-abelian theory: $S = \int_M \langle B \overset{\wedge}{,} dA + \frac{1}{2}[A, A]\rangle$ with $(A, B) \in \Omega^\bullet(M, \mathfrak{g})[1] \oplus \Omega^\bullet(M, \mathfrak{g}^*)[n-2]$.

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Cellular model:

- M – oriented cobordism.
- X – a cellular decomposition.
- $E = \mathrm{Ad}(\mathcal{P})$ – a G-local system in adjoint representation ($\mathcal{P}$ a flat G-bundle).
- Fields, symplectic structures, polarization, states - as in abelian theory.

Non-abelian action

Action:

$$S = \sum_{e\subset X-X_{\text{out}}} \sum_{k\geq 1} \frac{1}{k!} \langle B_{\varkappa(e)}, l_k^e(A|_{\bar{e}}, \dots, A|_{\bar{e}})\rangle \\ - i\hbar \sum_{e\subset X-X_{\text{out}}} \sum_{k\geq 2} \frac{1}{k!} q_k^e(A|_{\bar{e}}, \dots, A|_{\bar{e}}) + \langle B, A\rangle_{\text{in}}$$

where

- $l_k^{\bar{e}} := \sum_{e'\subset\bar{e}} (e')^* l_k^{e'} : \quad \wedge^k C^\bullet(\bar{e}, \mathfrak{g}) \to C^\bullet(\bar{e}, \mathfrak{g})$ are local L_∞ algebra operations on the complex of the closed cell $\bar{e}$.
- $q_k^e : \quad \wedge^k C^\bullet(\bar{e}, \mathfrak{g}) \to \mathbb{R}$ are the local unimodular (or "quantum") L_∞ operations on $\bar{e}$.
- $A|_{\bar{e}} = \sum_{e'\subset\bar{e}} (e')^* \cdot E(e > e') \circ A_{e'}$.

“Canonical” cellular non-abelian BF theory

Theorem

For X a CW complex, set $\mathsf{F} = C^\bullet(X,\mathfrak{g})[1] \oplus C_\bullet(X,\mathfrak{g}^*) \ \ni (A,\mathbf{B})$. There exist elements $\bar{S}_e \in \mathrm{Fun}(\mathsf{F})$, associated to cells $e \subset X$, of form

$$\bar{S}_e = \sum_{k\geq 1}\sum_{\Gamma_0}\sum_{e_1,\ldots,e_k\subset\bar{e}} \frac{C^e_{\Gamma_0,e_1,\ldots,e_k}}{|\mathrm{Aut}(\Gamma_0)|}\cdot\langle \mathbf{B}_e, \mathrm{Jacobi}_{\Gamma_0}(A_{e_1},\ldots,A_{e_k})\rangle$$
$$-i\hbar\sum_{k\geq 1}\sum_{\Gamma_1}\sum_{e_1,\ldots,e_k\subset\bar{e}} \frac{C^e_{\Gamma_1,e_1,\ldots,e_k}}{|\mathrm{Aut}(\Gamma_1)|}\cdot\mathrm{Jacobi}_{\Gamma_1}(A_{e_1},\ldots,A_{e_k})$$

with $C^e_{\Gamma,e_1,\ldots,e_k} \in \mathbb{R}$ – structure constants, such that

1. $S = \sum_{e\subset X}\bar{S}_e$ satisfies QME (not modified), $\Delta e^{\frac{i}{\hbar}S} = 0$.
2. $S = \langle \mathbf{B}, dA\rangle +$ higher corrections.
3. For e a point, $\bar{S}_e = \langle \mathbf{B}_e, \frac{1}{2}[A_e,A_e]\rangle$.

S is defined uniquely by properties (1-3), up to canonical transformation $S \sim S + \{S,R\} - i\hbar R$ with generator R satisfying the same ansatz.

“Canonical” cellular non-abelian BF theory

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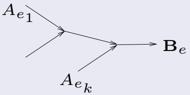
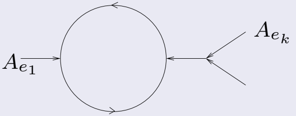
- Γ_0 runs over binary rooted trees



- Γ_1 runs over 3-valent 1-loop graphs



Some words on the proof. uL_∞ operations

Direction of the proof:

- Let $e_1, \ldots, e_N$ – cells of X in the order of non-decreasing dimension.
- We have a filtration $X_1 \subset X_2 \subset \cdots \subset X_N = X$, where $X_j = \cup_{i \leq j} e_i$.
- We go by induction in filtration, $X_j = X_{j-1} \cup e_j$.
  Note: while e_j are 0-dimensional, S_{X_j} is fixed by the condition $\bar{S}_e = \langle \mathbf{B}_e, \frac{1}{2}[A_e, A_e] \rangle$.
- We expand the solution of QME on X_{j-1} into $e = e_j$ (adjoining variables A_e, $\mathbf{B}_e$), order by order in A_e.
  Condition $S = \langle \mathbf{B}, dA \rangle + \cdots$ gives initial condition for the induction.

Remark: This is an AKSZ-like construction, inducing from value of theory on a point (∼target) and cell differential on X (∼source).

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Remark: This is an AKSZ-like construction, inducing from value of theory on a point (∼target) and cell differential on X (∼source).

Taylor expansion
$S = \sum_{k\geq 1} \frac{1}{k!} \langle \mathbf{B}, l_k(A, \ldots, A) \rangle - i\hbar \sum_{k \geq 2} q_k(A, \cdots, A)$ with
$l_k : \wedge^k C^\bullet(X, \mathfrak{g}) \to C^\bullet(X, \mathfrak{g})$, $\quad q_k : \wedge^k C^\bullet(X, \mathfrak{g}) \to \mathbb{R}$ – some multilinear operations on cochains.
QME $\Leftrightarrow$ quadratic relations on operations
– relations of a unimodular L_∞ algebra.

Back to cobordisms

Action:

$$S = \sum_{e \subset X - X_{\text{out}}} \sum_{k \geq 1} \frac{1}{k!} \langle B_{\varkappa(e)}, l_k^e(A|_{\bar{e}}, \ldots, A|_{\bar{e}}) \rangle - i\hbar \sum_{e \subset X - X_{\text{out}}} \sum_{k \geq 2} \frac{1}{k!} q_k^e(A|_{\bar{e}}, \ldots, A|_{\bar{e}}) + \langle B, A \rangle_{\text{in}}$$

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Back to cobordisms

Action:

$$S = \sum_{e\subset X-X_{\mathrm{out}}} \sum_{k\geq 1} \frac{1}{k!}\langle B_{\varkappa(e)}, l_k^e(A|_{\bar{e}},\ldots,A|_{\bar{e}})\rangle \\ - i\hbar \sum_{e\subset X-X_{\mathrm{out}}} \sum_{k\geq 2} \frac{1}{k!} q_k^e(A|_{\bar{e}},\ldots,A|_{\bar{e}}) + \langle B, A\rangle_{\mathrm{in}}$$

Boundary BFV action:

$$S_\partial(A_\partial, B_\partial) = \sum_{e\subset X_\partial} \sum_{k\geq 1} \frac{1}{k!}\langle B_{\varkappa\partial(e)}, l_k^e(A|_{\bar{e}},\ldots,A|_{\bar{e}})\rangle$$

Back to cobordisms

Action:

$$S = \sum_{e\subset X-X_{\mathrm{out}}} \sum_{k\geq 1} \frac{1}{k!}\langle B_{\varkappa(e)}, l_k^e(A|_{\bar{e}},\ldots,A|_{\bar{e}})\rangle$$

$$- i\hbar \sum_{e\subset X-X_{\mathrm{out}}} \sum_{k\geq 2} \frac{1}{k!} q_k^e(A|_{\bar{e}},\ldots,A|_{\bar{e}}) + \langle B, A\rangle_{\mathrm{in}}$$

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Differential on the space of states (quantum BFV charge) :

$$\Omega_\partial =$$

$$\sum_{e\subset X_{\mathrm{out}}} \sum_{k\geq 1} \frac{1}{k!}\left\langle l_k^e(\mathbb{A}|_{\bar{e}},\ldots,\mathbb{A}|_{\bar{e}}), -i\hbar\frac{\partial}{\partial \mathbb{A}_e}\right\rangle + \sum_{e\subset X_{\mathrm{in}}} \sum_{k\geq 1} \frac{1}{k!}\left\langle \mathbb{B}_{\varkappa_{\mathrm{in}}(e)}, l_k^e(\widehat{A}|_{\bar{e}},\ldots,\widehat{A}|_{\bar{e}})\right\rangle$$

where $\widehat{A}|_{\bar{e}} = -i\hbar \sum_{e'\subset\bar{e}} (e')^* \cdot E(e > e') \circ \frac{\partial}{\partial \mathbb{B}_{\varkappa_{\mathrm{in}}(e')}}$

Lemma

We have mQME on the level of fields:
$\left(-i\hbar\Delta_{\mathcal{V}} + \frac{i}{\hbar}\Omega_\partial\right) \circ \left(e^{\frac{i}{\hbar}S} \cdot \mu_\hbar^{\frac{1}{2}}\right) = 0$

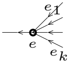
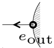
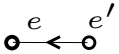
Follows from Theorem for “canonical” BF.

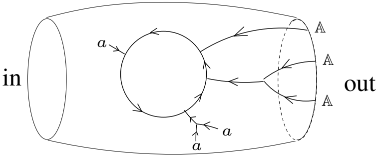
Residual fields $\mathcal{F}_{\text{res}} = H^\bullet(M, M_{\text{out}})[1] \oplus H^\bullet(M, M_{\text{in}})[n-2]$ and gauge-fixing $C^\bullet(X, X_{\text{out}}) \rightsquigarrow H^\bullet(M, M_{\text{out}})$ – as in abelian case.

Partition function

$$Z = \xi_{H^\bullet(M,M_{\text{out}})} \cdot \mu^{\frac{1}{2}}_{\mathcal{B}_\partial,\hbar} \cdot \tau(M, M_{\text{out}}) \cdot$$
$$\exp \frac{i}{\hbar} \sum_\Gamma \frac{(-i\hbar)^{\text{loops}(\Gamma)+\#V_q(\Gamma)}}{|\text{Aut}(\Gamma)|} \Phi_\Gamma(\mathbb{A},\mathbb{B};a,b)$$

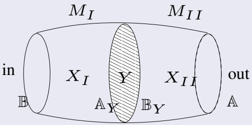
Feynman rules for Φ_Γ:

bulk $(k,0)$-vertex		$l^e_{k,e_1,\ldots,e_k}$
bulk $(k,1)$-vertex		$q^e_{k,e_1,\ldots,e_k}$
inward leaf		$i(a)_e$
outward leaf		$p^*(b)_{\varkappa(e)}$
out-boundary vertex		$\mathbb{A}_{e_{\text{out}}}$
in-boundary vertex		$\mathbb{B}_{\varkappa_{\text{in}}(e_{\text{in}})}$
edge		$\eta(e,\varkappa(e'))$



Theorem: non-abelian cellular BF is a quantum BV-BFV theory

1. mQME: $\left(-i\hbar\Delta_{\mathrm{res}} + \frac{i}{\hbar}\Omega_\partial\right) \circ Z = 0$.
2. Change of gauge-fixing induces a change $Z \to Z + \left(-i\hbar\Delta_{\mathrm{res}} + \frac{i}{\hbar}\Omega_\partial\right) \circ (\cdots)$.
3. Gluing: for $X = X_I \cup_Y X_{II}$



we have $\boxed{Z_X = P_*(Z_{X_I} *_Y Z_{X_{II}})}$ where $Z_{X_I} *_Y Z_{X_{II}} =$

$$\int_{\mathbb{A}_Y, \mathbb{B}_Y} Z_{X_I}(\mathbb{B}, \mathbb{A}_Y; a_I, b_I)\, \mathcal{D}^{\frac{1}{2}}\mathbb{A}_Y\, e^{-\frac{i}{\hbar}\langle \mathbb{B}_Y, \mathbb{A}_Y \rangle} \mathcal{D}^{\frac{1}{2}}\mathbb{B}_Y\, Z_{X_{II}}(\mathbb{B}_Y, \mathbb{A}; a_{II}, b_{II})$$

P_* – BV pushforward along $\mathcal{F}^I_{\mathrm{res}} \times \mathcal{F}^{II}_{\mathrm{res}} \to \mathcal{F}^{I\cup II}_{\mathrm{res}}$.

4. Z modulo $\left(-i\hbar\Delta_{\mathrm{res}} + \frac{i}{\hbar}\Omega_\partial\right) \circ (\cdots)$ is independent of X if X_{in}, X_{out} are fixed.

Remark: one can pass to the **reduced space of states** $H^\bullet_{\Omega_\partial}(\mathcal{H}_\partial) \cong H^\bullet_{CE}(H^\bullet(M_{\mathrm{out}}), \{l_k^{\mathrm{out}}\}) \otimes \left(H^\bullet_{CE}(H^\bullet(M_{\mathrm{in}}), \{l_k^{\mathrm{in}}\})\right)^*$, then Z^{reduced} is completely independent of X (including boundary).

Comments (on non-abelian model)

1. Structure constants in cellular action, $C^e_{\Gamma,e_1,\ldots,e_k}$, can be chosen to be rational.
2. Reduced space of states is an invariant of rational homotopy type of the boundary.
3. Z is an invariant of simple-homotopy type of M.
4. One can view $\mathcal{F}_X, e^{\frac{i}{\hbar}S_X} \cdot \mu^{\frac{1}{2}}_{X,\hbar}$, for different CW decompositions X of M, as different realizations of the quantum theory, compared by BV pushforwards along cellular aggregations. $\mathcal{F}_{\mathrm{res}}, Z$ is the minimal realization.
5. For X a "dense" CW decomposition of M, S_X approximates continuum non-abelian action $\int_M \langle B, dA + \frac{1}{2}[A,A]\rangle$.
6. Z_E arranges in a family over $\mathcal{M}_{\mathrm{loc}}(M) = \mathrm{Hom}(\pi_1(M),G)/G \ni E$. Z_E controls neighborhood of a singularity $E \in \mathcal{M}_{\mathrm{loc}}(M)$ and the behavior of R-torsion near the singularity.

Further programme

- Corners of $\mathrm{codim} \geq 2$, comparison with Baez-Dolan-Lurie extended TQFT formalism.
- Construct more general AKSZ-type cellular examples (e.g. 3D gravity with cosmological term) via cellular extension up from points.
- Compare cellular BF for $n = 3$ with Ponzano-Regge 3D quantum gravity (constructed via $6j$-symbols).
- Q-exact renormalization w.r.t. cellular aggregations and (hypothetical) connection to higher Igusa-Klein torsions.
- More general (than $\mathbb{A}$- and $\mathbb{B}$-) polarizations; Hitchin's connection comparing them infinitesimally.
- Kontsevich's deformation quantization of $\mathfrak{g}^*$ via cellular theory on a 2-disk.