Two-dimensional BF theory as a CFT

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Joint work with Andrey S. Losev and Donald R. Youmans,
arXiv:1712.01186, 1902.02738

Reminder. 2D BF theory:
Fix G a group, Σ - a surface.

- Action: $S_{cl} = \int_\Sigma \langle B, dA + \frac{1}{2}[A, A] \rangle$
- Fields: $A \in \Omega^1(\Sigma, \mathfrak{g}), \quad B \in \Omega^0(\Sigma, \mathfrak{g}^*)$

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Want to impose **Lorenz gauge** $\boxed{d * A = 0}$

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Faddeev-Popov (gauge-fixed) action

$$S = \int B\ dA + \lambda\ d * A + b\ d * dc$$

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$$\begin{array}{c}\underline{\mathsf{F}} = \underset{A}{\Omega^1} \oplus \underset{B,\lambda}{\underline{\mathbb{R}}^2} \oplus \underset{c}{\underline{\mathbb{R}}[1]} \oplus \underset{b}{\underline{\mathbb{R}}[-1]} \\ \downarrow \\ \Sigma\end{array}$$

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$$S = \int B\, dA + \lambda\, d*A + b\, d*dc \quad = S_{cl} + Q \underbrace{\int b\, d*A}_{\Psi - \text{g.f.fermion}}$$

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Gauge-fixed action in terms of complex fields

$$S = 2i \int_\Sigma -\gamma \bar{\boldsymbol{\partial}} \mathsf{a} + \bar{\gamma} \boldsymbol{\partial} \bar{\mathsf{a}} + b\, \boldsymbol{\partial} \bar{\boldsymbol{\partial}} c$$

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- Note: S is invariant under Weyl transformations of metric

Stress-energy tensor, BRST current

- **Stress-energy tensor:** $T_{\mu\nu} = \frac{-1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S = Q \underbrace{\left(\frac{-1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \Psi \right)}_{G_{\mu\nu}}$
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- Conservation laws: $$\begin{aligned} \bar{\partial} G \sim 0 \sim \partial \bar{G} \\ \bar{\partial} T \sim 0 \sim \partial \bar{T} \\ \bar{\partial} J \sim 0 \sim \partial \bar{J} \end{aligned}$$

Quantization (on $\mathbb{C}$)

Fix $\Sigma = \mathbb{C}$.

Correlators

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \frac{1}{Z} \int_{\text{fields}} e^{-\frac{1}{4\pi} S} \Phi_1(z_1) \cdots \Phi_n(z_n)$$

– given by Wick's lemma with propagators

$$\langle a(z)\gamma(w) \rangle = \frac{1}{z-w}, \quad \langle \bar{a}(z)\bar{\gamma}(w) \rangle = \frac{1}{\bar{z}-\bar{w}},$$
$$\langle c(z)b(w) \rangle = 2\log|z-w| + C$$

Composite fields:

$\Phi(z) \in \mathbb{F}_z = \{$differential polynomials in fields $a, \gamma, \bar{a}, \bar{\gamma}, b, c\}/\text{e.o.m.}$

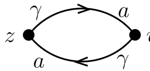
- $(\mathbb{F}_z, \cdot, Q)$ – free cdga, with $\cdot$ the **normally-ordered** product.
- $\mathbb{F}_z$ is freely generated by:

$$\underbrace{\partial^k b, \partial^k c, \partial^l a, \partial^l \gamma}_{\text{holom. sector}}; b, c; \underbrace{\bar{\partial}^k b, \bar{\partial}^k c, \bar{\partial}^l \bar{a}, \bar{\partial}^l \bar{\gamma}}_{\text{antiholom. sector}} \qquad \text{with } k \geq 1, \ l \geq 0$$

First examples of correlators and OPEs

An example of a correlator..

$$\langle :\gamma a:(z)\ \ :\gamma a:(w)\rangle =$$



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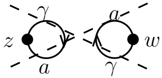

$$= \frac{-1}{(z-w)^2}$$

prohibited by normal ordering

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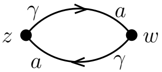

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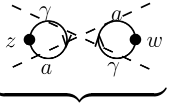

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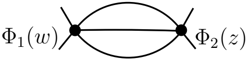
Operator product expansion (OPE):

$\Phi_1(w)\Phi_2(z) =$ sum of diagrams with external legs



Example of an OPE:

$$\underbrace{(\gamma\partial c)(w)}_{J(w)} \; \underbrace{(a\partial b)(z)}_{G(z)} \sim \frac{-1}{(w-z)^3} + \frac{:\gamma(w)a(z):}{(w-z)^2} - \frac{:\partial c(w)\,\partial b(z):}{w-z} + \text{reg}$$

$$\sim \frac{-1}{(w-z)^3} + \frac{:\gamma a:(z)}{(w-z)^2} + \frac{T(z)}{w-z} + \text{reg}$$

Some important OPEs

TT OPE

$$T(w)T(z) \sim \frac{0}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg}$$

Thus, central charge: $\boxed{c=0}$.

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part of S	$\gamma\bar{\partial}a$	$\bar{\gamma}\partial\bar{a}$	$b\,\partial\bar{\partial}c$
$(c,\bar{c})$	$(2,0)$	$(0,2)$	$(-2,-2)$

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$$T(w)G(z) \sim \frac{2\,G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg}, \quad T(w)J(z) \sim \frac{1\,J(z)}{(w-z)^2} + \frac{\partial J(z)}{w-z} + \text{reg}$$

I.e. G, J are **primary**, of dimensions $h_G = 2$, $h_J = 1$.

Mode operators

- $G_n : \Phi(z) \mapsto \oint_w \frac{dw}{2\pi i}(w-z)^{n+1} G(w)\Phi(z)$
- $L_n : \Phi(z) \mapsto \oint_w \frac{dw}{2\pi i}(w-z)^{n+1} T(w)\Phi(z)$ – Virasoro generators
- $Q^{\text{quantum}} : \Phi(z) \mapsto \frac{1}{4\pi} \oint_w J^{\text{tot}}(w)\Phi(z)$

Generators $\{G_n, L_n; \bar{G}_n, \bar{L}_n; Q\}$ span a super-Lie algebra with relations

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [L_n, G_m] = (n-m)G_{n+m}, \quad [Q, G_n] = L_n$$

Witten's descent

Descent equation

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)}$$

with $\mathcal{O}^{(p)} \in \underbrace{\mathbb{F}_z \otimes \wedge^p T_z^* \Sigma}_{\mathbb{F}_z^{(p)}}$

Solution of descent equation via G_{-1}

Set $\Gamma = -dz\, G_{-1} - d\bar{z}\, \bar{G}_{-1}$. Let $\mathcal{O}^{(0)}$ be Q-closed (a 0-observable), then

$$\mathcal{O}^{(0)} \;\rightarrow\; \mathcal{O}^{(1)} = \Gamma\mathcal{O}^{(0)} \;\rightarrow\; \mathcal{O}^{(2)} = \frac{1}{2}\Gamma^2\mathcal{O}^{(0)}$$

is a solution of descent. **Note:** $[Q, \Gamma] = d$.

Descent explicitly

- 0-observables in abelian BF: $\underbrace{H_Q(\mathbb{F}_z)}_{\mathbb{O}_z} = \{\text{polynomials in } B, c\}$
- In N-component theory, with $\underline{\mathsf{F}} \to \underline{\mathsf{F}} \otimes \underbrace{\mathbb{R}^N}_{V}$,

  $$\mathbb{O}_z = T^{\text{poly}}(V[1]) = \mathbb{C}[c^k, B_k = \text{``}\tfrac{\partial}{\partial c^k}\text{''}].$$

- $G_{-1}: \begin{array}{ccc} c & \mapsto & -a \\ \gamma & \mapsto & \partial b \\ \bar{\gamma}, b, a, \bar{a} & \mapsto & 0 \end{array}$ Also, G_{-1} acts as a derivation.
- Descent operator: $\Gamma: \begin{array}{ccc} c & \mapsto & A \\ B & \mapsto & -*db \end{array}$
- For $p(B, c)$ a 0-observable, the total descent

  $$\mathcal{O}^\bullet = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} = \boxed{p(B - *db, c + A)}$$

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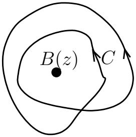
- **Remark:** $\mathcal{O}^\bullet = p(\widetilde{B}, \widetilde{A})|_{\mathcal{L}}$, where $\widetilde{B} = B + A^+ + c^+$, $\widetilde{A} = c + A + B^+$ – superfields of AKSZ formulation of BF, with $S^{\text{AKSZ}} = \int \widetilde{B} d\widetilde{A} + \lambda b^+$, and with $\mathcal{L}: \; A^+ = -*db, \; b^+ = d*A, \; B^+ = c^+ = \lambda^+ = 0$ the gauge-fixing Lagrangian.

Why care about descent?

- Topological correlators, e.g.

$$\langle B(z) \oint_C \underbrace{A(w)}_{\Gamma c} \rangle = 4\pi \, \mathrm{lk}(C, z)$$



- Deformations of CFTs $S \mapsto S + g \int \mathcal{O}^{(2)}$.
  - Example: $\mathcal{O}^{(0)} = \frac{1}{2} f^a_{bc} B_a c^b c^c$ – non-abelian deformation.
  - Example: $\mathcal{O}^{(0)} = W(c)$ an even polynomial of ghosts – (odd) Landau-Ginzburg superpotential deformation.
- BV algebra structure on $\mathbb{O}_z$:

$$(f, g) = \frac{1}{4\pi} \oint_w (\Gamma f)(w) \, g(z), \qquad G_0^- = \frac{1}{2i}(G_0 - \bar{G}_0)$$

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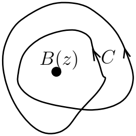

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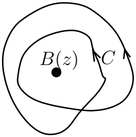

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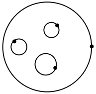
– It comes from the E_2^{fr}-algebra structure on $\mathbb{F}_z$ given by multi-OPE.



Abelian BF as a B-twisted $\mathcal{N} = (2,2)$-SCFT

Let $\boxed{j = \gamma a}$ – Noether current for $U(1)$ symmetry $a \mapsto e^{i\theta} a$, $\gamma \mapsto e^{-i\theta}\gamma$.

$\mathcal{N} = (2,2)$-superconformal model ("untwisted" abelian BF theory)

Untwist: $T \mapsto \boxed{T^{\mathrm{SUSY}} = T - \frac{1}{2}\partial j}$. Mode operators of $T^{\mathrm{SUSY}}, \underbrace{G, J}_{\text{supercharges}}, j$ satisfy $\mathcal{N} = 2$ super-Virasoro relations with $c = -3$.

	a	γ	b	c	G	J
h_{top}	1	0	0	0	2	1
h_{SUSY}	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{3}{2}$

Fields: $\underline{\mathsf{F}}^{\mathrm{SUSY}} = \underset{a}{K^{\otimes\frac{1}{2}}} \oplus \underset{\gamma}{K^{\otimes\frac{1}{2}}} \oplus \underset{\bar{a}}{\bar{K}^{\otimes\frac{1}{2}}} \oplus \underset{\bar{\gamma}}{\bar{K}^{\otimes\frac{1}{2}}} \oplus \underset{c}{\underline{\mathbb{R}}[1]} \oplus \underset{b}{\underline{\mathbb{R}}[-1]}$

Comparison of abelian BF to B model

Recall $\mathcal{N} = (2,2)$ supersymmetric sigma model (Landau-Ginzburg theory) with target $X = \mathbb{C}^N$:

$$S = \int d^2x \Big(\bar{\phi}_k \, \partial\bar{\partial}\phi^k - i\bar{\psi}_{+k} \, \bar{\partial}\psi_+^k - i\bar{\psi}_{-k} \, \partial\psi_-^k \Big)$$

$\pm$ - hol/anti-hol on Σ; $\Phi, \bar{\Phi}$ - hol/anti-hol on X.

Dictionary

abelian BF	coeff $V = \mathbb{R}^N$	c^k	b_k	$a^k, \bar{a}^k$	$\gamma^k, \bar{\gamma}^k$
B model	target $X = \mathbb{C}^N$	ϕ^k	$\bar{\phi}_k$	ψ_+^k, ψ_-^k	$\bar{\psi}_{+k}, \bar{\psi}_{-k}$

$$\begin{array}{ccc} \text{ab } BF & \xleftrightarrow{\text{parity}-\text{reversal}} & \text{B model} \\ \text{twist} \uparrow & & \uparrow \text{twist} \\ \text{untwisted ab } BF & & \text{LG} \end{array}$$

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- Theories on the left and right have different deformations due to different parity!

Towards Gromov-Witten invariants

Idea: $\langle \underbrace{G \cdots G}_{p} \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \to$ closed p-form ρ on $\mathcal{M}_{\Sigma,n}$.

$\int_C \rho$ – Gromov-Witten period on a p-cycle C.

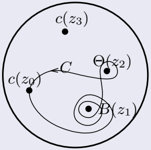
Toy example

$$\rho = \Big\langle \Gamma\Big(c(z_0)\ B(z_1)\ \underbrace{\Theta(z_2)}_{b\,\delta(\gamma)\delta(\bar\gamma)}\ c(z_3) \Big) \Big\rangle_{\mathbb{C}P^1} = 2d \arg \underbrace{\frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_2)(z_1 - z_3)}}_{\text{cross}-\text{ratio}}$$

$$\in \Omega^1_{cl}(\mathrm{Conf}_4(\mathbb{C}P^1))^{PSL_2(\mathbb{C})-\text{basic}} = \Omega^1_{cl}(\mathcal{M}_{0,4})$$

GW period:

$$\oint_{C \ni z_0} \rho = 4\pi\ \mathrm{lk}(C, [z_1] - [z_2])$$

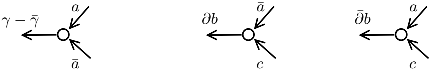


Non-abelian BF theory (as a deformation of abelian)

- $S = S_0 + g \int \mathcal{O}^{(2)}$,

$$\mathcal{O}^{(2)} = (-2) d^2 z \left(\langle \gamma - \bar{\gamma}, [a, \bar{a}] \rangle \quad + \langle \partial b, [\bar{a}, c] \rangle \quad + \langle \bar{\partial} b, [a, c] \rangle \right)$$

Feynman vertices :



- $Q = Q_0 + g\, Q_1$
- $T = T_0 + g \underbrace{T_1}_{\frac{1}{2}\langle \partial b, [a,c] \rangle} = Q(\underbrace{G}_{=G_0})$ — does not deform!

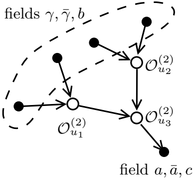
Non-abelian BF theory (as a deformation of abelian), cont'd

- E.o.m.: $$\begin{aligned} \bar\partial a - \tfrac{g}{2}[a,\bar a] &= 0 \\ \bar\partial\gamma + \tfrac{g}{2}[\bar a, \gamma - \bar\gamma] - \tfrac{g}{2}[c, \bar\partial b] &= 0 \\ \partial\bar\partial b + \tfrac{g}{2}[a, \bar\partial b] + \tfrac{g}{2}[\bar a, \partial b] &= 0 \\ \partial\bar\partial c + \tfrac{g}{2}\bar\partial[a,c] + \tfrac{g}{2}\partial[\bar a, c] &= 0 \end{aligned}$$ – don't preserve chiral sectors anymore!
- $J^{\mathrm{tot}} = 2i(dz\, J - d\bar z\, \bar J)$ with $J = J_0 + g \underbrace{J_1}_{\langle\gamma,[a,c]\rangle - \frac{1}{4}\langle\partial b,[c,c]\rangle}$.

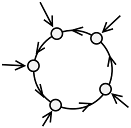
  $dJ^{\mathrm{tot}} \underset{e.o.m.}{\sim} 0$ but the chiral parts $J, \bar J$ are no longer conserved separately! $\to$ **No twisted $\mathcal{N} = (2,2)$ supersymmetry!**

Correlators

Correlators of fund. fields are finite sums of Feynman trees given by convergent[1] integrals



Loops vanish! (Boson-fermion cancellation in the loop.)

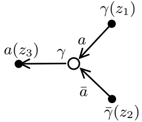


[1] Unless there is a bare b field $\rightarrow$ IR divergence

Examples of correlators

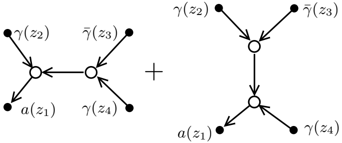
Example: 3-point function

$$\langle \gamma_a(z_1)\bar{\gamma}_b(z_2)a^c(z_3)\rangle = g f^c_{ab}\, \underbrace{\frac{1}{z_1-z_3}\log\left|\frac{z_1-z_2}{z_3-z_2}\right|}_{\int_{\mathbb{C}} \frac{d^2u}{2\pi}\frac{1}{(u-z_1)(\bar{u}-\bar{z}_2)(z_3-u)}}$$



Example: 4-point function

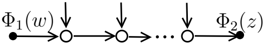
$$\langle a^a(z_1)\,\gamma_b(z_2)\,\bar{\gamma}_c(z_3)\,\gamma_d(z_4)\rangle = g^2\left(f^a_{be}f^e_{cd}\mathbb{I}_{1234} + f^a_{de}f^e_{cb}\mathbb{I}_{1432}\right),$$



$$\mathbb{I}_{1234} = \frac{1}{2z_{12}}\left(i\mathrm{D}\left(\frac{z_{34}}{z_{14}}\right) - i\mathrm{D}\left(\frac{z_{34}}{z_{24}}\right) + \log\left|\frac{z_{34}}{z_{14}}\right|\cdot\log\left|\frac{z_{23}}{z_{13}}\right| + \log\left|\frac{z_{14}}{z_{24}}\right|\cdot\log\left|\frac{z_{23}}{z_{34}}\right|\right),$$

$\mathrm{D}(z) = \operatorname{Im}\mathrm{Li}_2(z) + \arg(1-z)\log|z|$ – Bloch-Wigner dilogarithm

OPEs

OPE of fund. fields (or derivatives): 

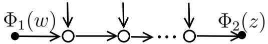
Examples:

$$a^a(w)\bar{\gamma}_b(z) \sim -g f^a_{bc} \boxed{\log|w-z|}\, a^c(z) + \mathrm{reg}$$

$$a^a(w)\gamma_b(z) \sim \frac{\delta^a_b}{w-z} + \frac{g}{2} f^a_{bc} \boxed{\frac{\bar{w}-\bar{z}}{w-z}}\, \bar{a}^c(z) + \mathrm{reg}$$

Here reg are continuous (not holomorphic) terms at $w \to z$.

OPEs

OPE of fund. fields (or derivatives): 

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Here reg are continuous (not holomorphic) terms at $w \to z$. E.g.,

$$c^a(w)b_b(z) \sim \delta^a_b 2\log|w-z| + \mathrm{reg}$$

$$\sim \delta^a_b 2\log|w-z| + g f^a_{bc} \log|w-z| \Big((w-z)a^c(z) + (\bar{w}-\bar{z})\bar{a}^c(z)\Big) + \underbrace{\mathrm{reg}^{(1)}}_{C^1 \text{ at } w\to z}$$

Composite fields

Composite fields are built via renormalized products

$$(\Phi_1\Phi_2)(z) := \lim_{w\to z}\left(\Phi_1(w)\Phi_2(z) - \underbrace{\Big[\Phi_1(w)\Phi_2(z)\Big]_{\text{sing}}}_{\text{sing part of the OPE}}\right)$$

- Sing. part of the OPE is of the form

$$\Big[\Phi_1(w)\Phi_2(z)\Big]_{\text{sing}} = \sum_{p,q,r}(w-z)^{-p}(\bar{w}-\bar{z})^{-q}\log^r|w-z|\;\Phi_{pqr}(z)$$

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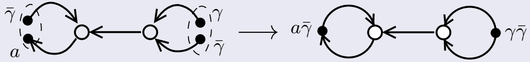
  – It is defined relative to the local coordinate z.
- Composite fields built as $\Phi_1\cdots\Phi_n$ depend on the order of merging of factors.
- Examples composite fields: T, G, J^{tot}, e.o.m. (when endowed with merging-order data). $\bar\partial T$, $\bar\partial G$, dJ^{tot} and e.o.m. vanish under the correlator.

Examples of correlators and OPEs of composite fields

Example

$$\left\langle (a^a\bar{\gamma}_b)(w)(\gamma_c\bar{\gamma}_d)(z)\right\rangle = g^2 f^a_{be} f^e_{cd} \frac{\log^2|w-z|}{w-z}$$



Example

$$\langle (a^a\bar{\gamma}_{b_1}\cdots\bar{\gamma}_{b_n})(w)\gamma_c(z)\rangle = \frac{g^n}{n!}\left(\sum_{\sigma\in S_n} f^a_{b_{\sigma(1)}e_1}\cdots f^{e_{n-1}}_{b_{\sigma(n)}c}\right)\frac{\log^n|w-z|}{w-z}$$

Example of an OPE

$$(a^a\bar{\gamma}_b)(w)\,\bar{\gamma}_c(z) \sim$$
$$-g f^a_{ce}\log|w-z|(a^e\bar{\gamma}_b)(z) - \frac{g}{2}(f^a_{be}f^e_{cf} - f^a_{ce}f^e_{bf})\log^2|w-z|a^f(z) + \text{reg}$$

Non-abelian BF as a TCFT

- $T(z) = \underbrace{\frac{1}{4\pi} \oint_w J^{\text{tot}}(w)}_{Q^{\text{quantum}}} G(z)$
- $T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg}$ - corresponds to $c = 0$ CFT.
- $T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg}$
- Fund. fields $a, \bar{a}, \gamma, \bar{\gamma}, b, c$ are all primary, with same dimensions as in the abelian case.

Vertex operators

Fix $u \in \mathfrak{g}$; fix $v \in \mathfrak{g}$ – eigenvector of ad_u with eigenvalue α; $\rho \in \mathfrak{g}^*$ – eigenvector of ad^*_u with eigenvalue $-\alpha$.

Examples of fields with a quantum correction to dimension ("vertex operators")

Field $V_{u,\rho} := \langle \rho, a \rangle e^{\langle u, \bar{\gamma} \rangle}$ is primary, of dimension $(h = 1 - \frac{\alpha g}{2}, \bar{h} = -\frac{\alpha g}{2})$.
Field $W_{u,v} := \langle v, \gamma - \bar{\gamma} \rangle e^{\langle u, \bar{\gamma} \rangle}$ is primary, of dimension $(h = \frac{\alpha g}{2}, \bar{h} = \frac{\alpha g}{2})$

Dimension (e.g. for V) can be seen from:

- TV OPE computation.
- Coordinate dependence (under scaling) of the singular subtractions in V.
- Fields $\boxed{V_n = \frac{\varkappa^{-n}}{n!} \langle \rho, a \rangle \cdot \langle u, \bar{\gamma} \rangle^n}$, $n \geq 0$, with $\varkappa = -\frac{g\alpha}{2}$, form an infinite Jordan block of L_0: $L_0 V_n = V_n + V_{n-1}$. Therefore, $V = \sum_{n=0}^{\infty} \varkappa^n V_n$ has dimension $h = 1 + \varkappa$.

Example of a correlator: $\langle V(w)\, \langle v, \gamma(z) \rangle \rangle = \langle \rho, v \rangle \frac{|w - z|^{\alpha g}}{w - z}$

Summary

1. Gauge-fixed BF theory is a TCFT.
2. Abelian theory is a B-twisted $\mathcal{N} = (2, 2)$ supersymmetric CFT with an odd target. Its deformations are different from the even case (B model).
3. Non-abelian theory has answers given by convergent integrals ($\rightarrow$ polylogarithms).
4. OPEs of non-abelian theory contain expressions $z^{-p}\bar{z}^{-q}\log^r|z|$.
5. One has infinite Jordan blocks for L_0 and fields with anomalous dimension.

Where we want to go with this:

- Gromov-Witten invariants coming from BF. WDVV equation?
- $BF \rightarrow$ AKSZ as a CFT.
- Understand better the delta-function observables necessary to define the theory on $\mathbb{C}P^1$.

References: arXiv:1712.01186, 1902.02738.