Two-dimensional perturbative scalar field theory with polynomial potential and cutting-gluing

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Zoom Seminar: Differential Geometry and Mathematical Physics
June 12, 2020

Joint work with Santosh Kandel and Konstantin Wernli,
arXiv:1912.11202

Atyiah-Segal's axiomatics of QFT

n-dimensional QFT (in the sense of Atiyah-Segal)

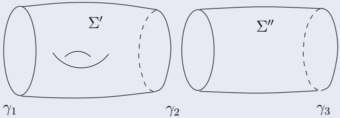
Data:

1. Closed $(n-1)$-manifold γ $\rightarrow$ $\mathcal{H}_\gamma$ – "space of states" (Hilbert space).
2. n-manifold Σ with boundary $\rightarrow$ $Z_\Sigma \in \mathcal{H}_{\partial\Sigma}$ – "partition function."

Axioms:

a. Multiplicativity $\mathcal{H}_{\gamma_1 \sqcup \gamma_2} = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$
b. Gluing: if $\partial\Sigma' = \gamma_1 \sqcup \gamma_2$, $\partial\Sigma'' = \gamma_2 \sqcup \gamma_3$ and $\Sigma = \Sigma' \cup_{\gamma_2} \Sigma''$, then

$$\boxed{Z_\Sigma = \langle Z_{\Sigma'}, Z_{\Sigma''}\rangle_{\mathcal{H}_{\gamma_2}}}$$



Example: quantum mechanics

Fix $\mathcal{H}$ a Hilbert space, $\widehat{H}$ a self-adjoint operator.

- $\mathrm{pt} \mapsto \mathcal{H}$
- interval of length $t \mapsto Z_t = e^{-\frac{i}{\hbar}\widehat{H}t} \quad \in \mathrm{End}(\mathcal{H}) = \mathcal{H}^* \widehat{\otimes} \mathcal{H}$

Atyiah-Segal's axiomatics of QFT: functorial language

Another formulation:

$(\mathcal{H}, Z):$	Cob_n	$\to$	Vect
Ob	$(n-1)$-manifold γ	$\mapsto$	$\mathcal{H}_\gamma$ – vector space
Mor	n-cobordism $\gamma_1 \xrightarrow{\Sigma} \gamma_2$	$\mapsto$	$Z_\Sigma : \mathcal{H}_{\gamma_1} \to \mathcal{H}_{\gamma_2}$ – linear map
$\circ$	sewing	$\mapsto$	composition $Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$
$\otimes$	$\sqcup$	$\mapsto$	$\otimes$

– a functor of symmetric monoidal categories.

Scalar field theory (as a classical field theory)

Action:

$$S_\Sigma(\phi) = \int_\Sigma \frac{1}{2} d\phi \wedge * d\phi + \frac{m^2}{2}\phi^2 d\,\mathrm{vol} + p(\phi) d\,\mathrm{vol}$$

Here:

- Σ - 2d surface with a Riemannian metric
- field $\phi \in C^\infty(\Sigma)$
- $m > 0$ mass
- $p(\phi) = \sum_{n \geq 3} \frac{p_n}{n!}\phi^n$ – polynomial interaction potential

Critical point equation:

$$\delta S = 0 \quad \Leftrightarrow \quad (\Delta + m^2)\phi + p'(\phi) = 0$$

Path integral on a closed surface

For Σ closed, the partition function is:

$$Z_\Sigma = \text{“} \int_{C^\infty(\Sigma)} \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S_\Sigma(\phi)} \text{”} := \det{}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \frac{\hbar^{E-V}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma$$

Here:

1. $\det(\Delta + m^2) = e^{-\zeta'(0)}$ with $\zeta(s) = \sum_\lambda \lambda^{-s}$ – zeta-regularized determinant

Examples:

- For S^1, $\det(\Delta + m^2) = 4 \sinh^2 \pi m R$.
- For S^2,

$$\det(\Delta + m^2) = e^{\frac{1}{2} - 4\zeta'(-1)} R^{-2(\frac{1}{3} - m^2 R^2)} \cdot \frac{\pi e^{-2m^2 R^2}}{\cos \pi (\frac{1}{4} - m^2 R^2)^{\frac{1}{2}}} \cdot \mathbb{G}(\alpha_1)^2 \mathbb{G}(\alpha_2)^2$$

where $\alpha_{1,2}$ – roots of $\alpha^2 - \alpha + (mR)^2 = 0$ and $\mathbb{G}$ – Barnes' double Gamma function.

Path integral on a closed surface

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Here:

1. $\det(\Delta + m^2) = e^{-\zeta'(0)}$ with $\zeta(s) = \sum_\lambda \lambda^{-s}$ – zeta-regularized determinant
2. $\hbar$ - formal infinitesimal parameter
3. Γ runs over graphs
4. Evaluation of a graph:

$$\Phi_\Gamma := \int_{\mathrm{Conf}_V(\Sigma)\ni(x_1,\ldots,x_V)} \prod_{\text{vertices } v} (-p_{\mathrm{val}(v)}) \cdot \prod_{\text{edges } e=(uw)} G(x_u, x_w)\, d^2x_1 \cdots d^2x_V$$

where $G(x,y)$ – Green's function for $\Delta + m^2$.

Note: Coefficient of $\hbar^k$, $k \geq 0$, is a sum of **finitely many** graphs.

Short loops in graphs

Note:

- $G(x,y) \underset{x\to y}{\sim} -\frac{1}{2\pi}\log d(x,y)$. So, integrals in Φ_Γ are convergent, **except if** Γ **contains a short loop** – $G(x,x)$ needs to be regularized.
- Naive presciption: $G(x,x) := 0$. - Incompatible with gluing.
- Better solution $\boxed{G(x,x) := \tau(x)}$ – "tadpole function," a datum of regularization of the theory.

Qunatization with boundary

Let $\gamma = S^1 \sqcup \cdots \sqcup S^1$ – a circle, or several circles, endowed with Riemannian metric.

Space of states

Naively: $\mathcal{H}_\gamma = \mathrm{Fun}(C^\infty(\gamma))$. More explicitly:

$$\mathcal{H}_\gamma = \left\{ \Psi(\eta) = \sum_{n\geq 0} \int_{\mathrm{Conf}_n(\gamma)} \psi_n(x_1, \ldots, x_n)\eta(x_1)\cdots\eta(x_n)dx_1\cdots dx_n \right\}$$
$$= \bigoplus_{n\geq 0} \mathcal{H}_\gamma^{(n)}$$

where

- $\eta \in C^\infty(\gamma)$ – boundary field
- $\psi_n \in C^\infty(\mathrm{Conf}_n(\gamma))$ – "n-particle wavefunction," assumed to have "admissible singularities" on diagonals:
  - $\psi_n = O(\log d(x_i, x_j))$ if $x_i \to x_j$.
  - $\psi_n = O(\epsilon^{2-k})$ if $k \geq 3$ points coalesce at mutual distances $O(\epsilon)$.

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Qunatization with boundary, cont'd

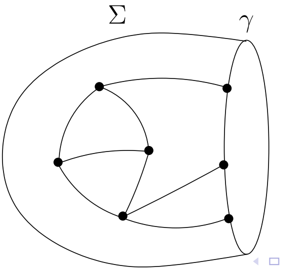
Partition function for a surface Σ with boundary γ:

$$Z_\Sigma(\eta) = \text{“} \int_{\phi|_\gamma = \sqrt{\hbar}\,\eta} \mathcal{D}\phi\, e^{-\frac{1}{\hbar} S_\Sigma(\phi)} \text{”}$$

$$:= e^{-\frac{1}{2}\int_\gamma \eta D_\Sigma(\eta)} \det{}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \frac{\hbar^{E-V-\frac{n}{2}}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma(\eta)$$

Here:

- Γ runs over graphs with 1-valent boundary vertices allowed; no $\gamma - \gamma$ edges; $n = \#\gamma\text{−vertices}$, $V = \#\text{bulk vertices}$.



Qunatization with boundary, cont'd

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Here:

- Γ runs over graphs with 1-valent boundary vertices allowed; no $\gamma - \gamma$ edges; $n = \#\gamma\text{−vertices}$, $V = \#\text{bulk vertices}$.
- Feynman rules for $\Phi_\Gamma(\eta)$:

boundary vertices	$\to$	$\eta(x_i)$
bulk vertices	$\to$	$-p_{\mathrm{val}(v)}$
bulk-bulk edges	$\to$	$G(y_\alpha, y_\beta)$
		– Green's fun. with Dirichlet b.c. on γ
bulk-boundary edges	$\to$	$\frac{\partial}{\partial n(x_i)} G(x_i, y_\alpha)$
		– normal derivative of Green's fun.

Take a product of decorations and integrate over $y_\alpha \in \Sigma$, $x_i \in \gamma$.

Qunatization with boundary, cont'd

Partition function for a surface Σ with boundary γ:

$$Z_\Sigma(\eta) = \text{``} \int_{\phi|_\gamma = \sqrt{\hbar}\,\eta} \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_\Sigma(\phi)} \text{''}$$

$$:= e^{-\frac{1}{2}\int_\gamma \eta D_\Sigma(\eta)} \det{}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \frac{\hbar^{E-V-\frac{n}{2}}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma(\eta)$$

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$$\Phi_\Gamma(\eta) = \int_{\mathrm{Conf}_n(\gamma)} dx_1 \cdots dx_n \prod_i \eta(x_i) \cdot \int_{\mathrm{Conf}_V(\Sigma)} d^2y_1 \cdots d^2y_V \cdot$$

$$\cdot \prod_\alpha (-p_{\mathrm{val}(v_\alpha)}) \cdot \prod_{(\alpha\beta)\in E} G(y_\alpha, y_\beta) \cdot \prod_{(i\alpha)\in E} \frac{\partial}{\partial n(x_i)} G(x_i, y_\alpha)$$

Qunatization with boundary, cont'd

Partition function for a surface Σ with boundary γ:

$$Z_\Sigma(\eta) = \text{“} \int_{\phi|_\gamma = \sqrt{\hbar}\,\eta} \mathcal{D}\phi\; e^{-\frac{1}{\hbar} S_\Sigma(\phi)} \text{”}$$

$$:= e^{-\frac{1}{2}\int_\gamma \eta D_\Sigma(\eta)} \det{}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \frac{\hbar^{E-V-\frac{n}{2}}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma(\eta)$$

Here:

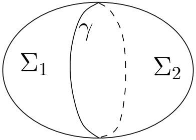
- Γ runs over graphs with 1-valent boundary vertices allowed; no $\gamma - \gamma$ edges; $n = \#\gamma\mathrm{-vertices}$, $V = \#\mathrm{bulk\ vertices}$.
- $\det(\Delta + m^2)$ – zeta-regularized determinant of $\Delta + m^2$ with Dirichlet b.c.
- $D_\Sigma : \underbrace{\eta}_{\in C^\infty(\gamma)} \mapsto \phi_\eta \mapsto \partial_n \phi_\eta|_\gamma$ – Dirichlet-to-Neumann operator.

  ϕ_η – a solution of $(\Delta + m^2)\phi = 0$ with b.c. $\phi|_\gamma = \eta$.

$$\boxed{Z_\Sigma(\eta) \in \mathcal{H}_\gamma}$$ (slightly cheating here)

Gluing

Let $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ a closed surface.



Gluing I. Formal gluing: we expect

$$Z_\Sigma = \text{“} \int_{C^\infty(\gamma)} \mathcal{D}\eta \; Z_{\Sigma_1}(\eta) Z_{\Sigma_2}(\eta) \text{”}$$

– “Fubini theorem for path integrals.”

Gluing II. Introduce $\widehat{Z}_{\Sigma_i} = \det_{\Sigma_i}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \cdots$, such that $Z_{\Sigma_i} = e^{-\frac{1}{2}\int_\gamma \eta D_{\Sigma_i}(\eta)} \widehat{Z}_{\Sigma_i}$.
Also, introduce an L^2 pairing $\langle , \rangle_D : \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma \to \mathbb{R}$,

$$\langle \underbrace{\Psi_1}_{\in \mathcal{H}_\gamma^{(n)}}, \underbrace{\Psi_2}_{\in \mathcal{H}_\gamma^{(m)}} \rangle_D = \sum_\xi \int_{\mathrm{Conf}_{n+m}(\gamma)} \psi_1(x_1, \ldots, x_n)\, \psi_2(x_{n+1}, \ldots, x_{n+m}) \cdot$$
$$\cdot \prod_{(i,j)\in\xi} D^{-1}(x_i, x_j) \cdot \det{}^{-\frac{1}{2}}(D)$$

Here:

- $D = D_{\Sigma_1} + D_{\Sigma_2}$,
- ξ runs over perfect matching on a set of $n+m$ elements

Theorem A

$$Z_\Sigma = \langle \widehat{Z}_{\Sigma_1}(\eta), \widehat{Z}_{\Sigma_2}(\eta) \rangle_D$$

R.h.s. is a rewriting of " $\int_{C^\infty(\gamma)} \mathcal{D}\eta\, e^{-\frac{1}{2}\int_\gamma \eta D(\eta)} \widehat{Z}_{\Sigma_1}(\eta)\, \widehat{Z}_{\Sigma_2}(\eta)$ " by Wick's lemma.

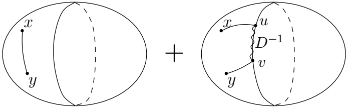
Idea of proof of Theorem A

- (a) Burghelea-Friedlander-Kappeler gluing formula for zeta-regularized determinants:

$$\det_{\Sigma}(\Delta + m^2) = \det_{\Sigma_1}(\Delta + m^2) \cdot \det_{\Sigma_2}(\Delta + m^2) \cdot \det_{\gamma} D$$

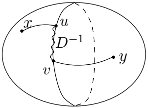
- (b) Gluing of Green's functions. For $x, y \in \Sigma_1$:

$$G_{\Sigma}(x, y) = G_{\Sigma_1}(x, y) + \int_{\gamma} du \int_{\gamma} dv \frac{\partial}{\partial n(u)} G_{\Sigma_1}(x, u) D^{-1}(u, v) \frac{\partial}{\partial n(v)} G_{\Sigma_1}(v, y)$$



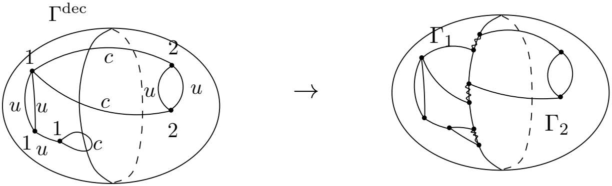
For $x \in \Sigma_1, y \in \Sigma_2$:

$$G_{\Sigma}(x, y) = \int_{\gamma} du \int_{\gamma} dv \frac{\partial}{\partial n(u)} G_{\Sigma_1}(x, u) D^{-1}(u, v) \frac{\partial}{\partial n(v)} G_{\Sigma_2}(v, y)$$



Idea of proof of Theorem A, cont'd

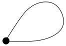
$$\Phi_\Gamma^\Sigma = \sum_{V_\Gamma \to \{1,2\}, E_\Gamma \to \{u,c\}} \Phi_{\Gamma^{\mathrm{dec}}}^\Sigma = \sum_{\Gamma_1,\Gamma_2} \langle \Phi_{\Gamma_1}^{\Sigma_1}, \Phi_{\Gamma_2}^{\Sigma_2} \rangle_D$$



Here

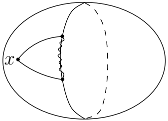
- u, c ("uncut", "cut") – terms in the gluing formula for Green's functions.
- $1, 2$ – restrict integration to Σ_1, Σ_2.

Tadpoles

In fact, $Z_\Sigma(\eta) = Z_\Sigma^\tau(\eta)$, with $\tau \in C^\infty(\Sigma)$ – a tadpole function assigned to 

Then $Z_\Sigma^\tau = \langle \widehat{Z}_{\Sigma_1}^{\tau_1}, \widehat{Z}_{\Sigma_2}^{\tau_2} \rangle_D$ with $\tau = \tau_1 * \tau_2$ glued tadpole:

$$\tau(x) \underset{x\in\Sigma_1}{=} \tau_1(x) +$$



In particular, for $\tau_1 = 0, \tau_2 = 0$, $\tau_1 * \tau_2 \neq 0$!
So, assigning $\tau = 0$ for all surfaces is incompatible with gluing.

A natural choice compatible with gluing.

Zeta-regularized tadpole

$$\tau(x) = \tau_{\zeta-\mathrm{reg}}(x) = \lim_{s\to 1}\left(\int_0^\infty dt\, t^{s-1} K(t,x,x) - \frac{1}{4\pi(s-1)}\right)$$

with $K(t,x,y)$ the heat kernel for $\Delta + m^2$.

Another choice:

$$\tau_{\mathrm{split}}(x) = \lim_{y\to x}\left(G(x,y) + \frac{1}{2\pi}\log d(x,y)\right) \qquad = \tau_{\zeta-\mathrm{reg}}(x) + \frac{\log 2 - \gamma}{2\pi}$$

One might ask an additional compatibility condition:

$$\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}\int_\Sigma \frac{1}{2}d\phi\wedge * d\phi + \frac{m^2}{2}\phi^2 d\,\mathrm{vol} + \frac{\alpha}{2}\phi^2 d\,\mathrm{vol}}$$

$$= \left(1 - \frac{1}{2}\int_\Sigma \alpha(x)\tau(x) d^2x + O(\alpha^2)\right)\cdot \det{}^{-\frac{1}{2}}(\Delta + m^2)$$

$$= \det{}^{-\frac{1}{2}}(\Delta + m^2 + \alpha)$$

This fixes the tadpole uniquely: $\tau = \tau_{\zeta-\mathrm{reg}}$.

Gluing revisited

Drawback of Gluing II: **non-functoriality**. Pairing $\langle , \rangle_D$ depends on Σ_1, Σ_2. Cannot use it to define a functor $\mathrm{Cob}_2^{\mathrm{Riem}} \to \mathrm{Hilb}$.

Gluing III.

- Fact: D_Σ is "very close" to the operator $\varkappa = \sqrt{\Delta + m^2}|_\gamma$. In fact: $s = D_\Sigma - \varkappa$ is a pseudodifferential operador of order ≤ -2.

Examples:

	λ_n	$\lambda_n - \omega_n$
disk	$m\frac{I'_n(mR)}{I_n(mR)}$	$-\frac{m^2 R}{2} n^{-2} + O(n^{-3})$
hemisphere	$\frac{2}{R}\frac{\Gamma(\frac{n+1+\alpha_1}{2})\Gamma(\frac{n+1+\alpha_2}{2})}{\Gamma(\frac{n+\alpha_1}{2})\Gamma(\frac{n+\alpha_2}{2})}$ $\alpha^2 - \alpha + (mR)^2 = 0$	$-\frac{m^2 R}{4} n^{-3} + O(n^{-4})$
cylinder	$\omega_n \coth H\omega_n$	$O(n^{-\infty})$

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- Fact: D_Σ is "very close" to the operator $\varkappa = \sqrt{\Delta + m^2}|_\gamma$. In fact: $s = D_\Sigma - \varkappa$ is a pseudodifferential operador of order ≤ -2.
- Let $\overline{Z}_\Sigma(\eta) \in e^{\frac{1}{2}\int_\gamma \eta\varkappa(\eta)} Z_\Sigma(\eta) \quad \in \mathcal{H}_\gamma$.
- One has the gluing formula:

$$\boxed{\overline{Z}_\Sigma = \langle \overline{Z}_{\Sigma_1}, \overline{Z}_{\Sigma_2} \rangle_\varkappa}$$

  where $\langle , \rangle_\varkappa$ is an L^2 pairing on $\mathcal{H}_\gamma$ defined similarly to $\langle , \rangle_D$, replacing $D \to 2\varkappa$.
- **Remark:** $\langle , \rangle_\varkappa$ can be defined using an ∞-dim integral against Gaussian measure $\mu_\varkappa$ on $\mathcal{D}'(\gamma)$.
  Measures μ_D and $\mu_\varkappa$ are *equivalent* (since $D - 2\varkappa$ is small enough), with Radon-Nikodym derivative $\frac{\mu_D}{\mu_\varkappa} = e^{-\frac{1}{2}\int_\gamma \eta(D-2\varkappa)(\eta)}$

Functoriality

Theorem B

$(\mathcal{H}, \overline{Z})$ is a functor of symmetric monoidal categories,

$(\mathcal{H}, \overline{Z})$:	$\mathrm{Cob}_2^{\mathrm{Riem}}$	$\to$	Hilb
Ob	collared Riem 1-manifolds	$\mapsto$	real Hilbert spaces $\otimes \mathbb{R}[[\hbar^{\frac{1}{2}}]]$
Mor	Riem 2-cobordisms	$\mapsto$	Hilbert-Schmidt operators
$\circ$	gluing	$\mapsto$	composition of lin operators
$\otimes$	$\sqcup$	$\mapsto$	$\widehat{\otimes}$

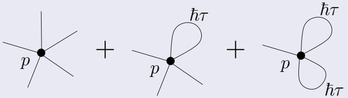
Aside: Tadpoles vs RG flow

Petal diagram resummation

$$\boxed{Z^{\tau,p} = Z^{0,\widetilde{p}}},\qquad p(\phi) \text{ – interaction polynomial, } \tau \text{ – tadpole}$$

where

$$\widetilde{p}(\phi) =$$



$$+\cdots \quad =: \mathcal{R}_\tau(p)$$

Properties:

- $\mathcal{R}_{\tau_1} \circ \mathcal{R}_{\tau_2} = \mathcal{R}_{\tau_1+\tau_2}$
- $Z^{\tau_1,p} = Z^{\tau_2,q}$ if $q = \mathcal{R}_{\tau_1-\tau_2}(p)$.
- $\mathcal{R}_\tau(p) = p_\tau$ satisfies the RG flow equation

$$\left(\frac{\partial}{\partial\tau} - \frac{\hbar}{2}\frac{\partial^2}{\partial\phi^2}\right) p_\tau = 0$$

- Example of a solution: $p_\tau(\phi) = \underbrace{e^{\frac{\hbar^2}{2}\tau\alpha^2}}_{\text{multiplicative scaling}} \cdot \underbrace{e^{\alpha\phi}}_{p(\phi)}$.

Aside: Tadpoles vs RG flow cont'd

Application:

$$Z^{\tau,p} = Z^{\tau_\Lambda, p_\Lambda}$$

where

- $p(\phi)$ some fixed polynomial
- $\tau = \tau_{\zeta-\mathrm{reg}}$ – *renormalized* tadpole
- $\tau_\Lambda = G_\Lambda(x,x) = \int_{1/\Lambda^2}^\infty dt\, K(t,x,x) \underset{\Lambda\to\infty}{\approx} \tau_{\zeta-\mathrm{reg}} + \frac{\log\Lambda}{2\pi} - \frac{\gamma}{4\pi}$ – just *regularized* tadpole (via proper time cut-off)
- $p_\Lambda = \mathcal{R}_{\tau-\tau_\Lambda}(p) = \mathcal{R}_{-\frac{\log\Lambda}{2\pi}+\frac{\gamma}{4\pi}}(p)$ – potential *with counterterms* introduced to make Z finite at $\Lambda\to\infty$

$$\frac{\partial}{\partial\log\Lambda} p_\Lambda(\phi) = -\frac{\hbar}{4\pi}\frac{\partial^2}{\partial\phi^2} p_\Lambda(\phi)$$

– RG flow equation for the action with counterterms.

Appendix: trace anomaly

- Trace of classical stress-energy tensor:

$$\operatorname{tr} T_{cl}(x) = \frac{-2}{\sqrt{\det g}} \frac{\delta}{\delta\sigma}\Big|_{\sigma=0} S_\Sigma^{g\to e^\sigma g} \quad = \quad -m^2\phi^2 - p(\phi)$$

- Trace of quantum stress-energy tensor:

$$\langle \operatorname{tr} T_q(x)\rangle = \hbar \frac{2}{\sqrt{\det g}} \frac{\delta}{\delta\sigma}\Big|_{\sigma=0} \log Z_\Sigma^{g\to e^\sigma g}$$

"Trace anomaly"

$$\langle \operatorname{tr} T_q(x)\rangle = \left\langle \operatorname{tr} T_{cl}(x) + \underbrace{\frac{\hbar}{4\pi}\left(\frac{R(x)}{6} - m^2 - \frac{\partial^2}{\partial\phi^2}p(\phi)\right)}_{\text{trace anomaly}} \right\rangle$$

where $R(x)$ is the scalar curvature.