Torsion as an Integral

IDEA: WANT TO write $\underbrace{\tau(X)}_{\text{manifold or CW complex}}$ - torsion as a field theory-type integral on X as a "space-time"

$$\int_{\mathcal{F}_X} e^{\frac{i}{\hbar} S_X(\varphi)} \, \mathcal{D}\varphi$$

φ – field on X

Algebraic story:

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n$$

cochain complex of vector spaces $/\mathbb{R}$ with inner product $(,)$ on C^j

Assume $(C^\bullet, d)$ acyclic. Choose a chain contraction K - section

$$\begin{tikzcd} C^j \arrow[r, "d"] & C^{j+1}_{exact} \arrow[l, bend left, dashed, "K"] \end{tikzcd}$$

Thus: $C^j = \underbrace{C^j_{ex}}_{im(d)} \oplus \underbrace{C^j_{coex}}_{im(K)}$

$$\begin{tikzcd} & C^1_{ex} & C^2_{ex} & & C^n_{ex} \\ C^0_{coex} \arrow[ur, "d"] & C^1_{coex} \arrow[ur, "d"] \arrow[ul, dashed] & C^2_{coex} \arrow[ul, dashed] & \cdots \arrow[ur] & \end{tikzcd}$$

Torsion: $\tau(C^\bullet, d, (,)) =$ "Sdet(d)" $= \prod_{j=0}^{n-1} \left(\det_{C^j_{coex} \to C^{j+1}_{ex}}(d)\right)^{(-1)^j}$

$$= \prod_{j=0}^{n} \left(\det_{C^j} \Delta\right)^{-\frac{(-1)^j \cdot j}{2}} \in \mathbb{R}_{>0} \quad \text{where } \Delta = d d^T + d^T d : C^j \to C^j$$

τ does not depend on K

Non-acyclic case: choose an (i, p, K) triple

$$\begin{tikzcd} C^\bullet \arrow[loop above, "K"] \arrow[d, shift left, "p"] \\ H^\bullet \arrow[u, shift left, "i"] \end{tikzcd}$$

$p \circ i = id$

$d i = p d = 0$

$\boxed{dK + Kd = id - i \circ p}$

$K^2 = 0$

$K i = p K = 0$

Hodge decomposition: $C^j = \underbrace{C^j_{harm}}_{im(i)} \oplus \overbrace{\underbrace{C^j_{ex}}_{im(d)} \oplus \underbrace{C^j_{coex}}_{im(K)}}^{\ker(p)}$

$$\boxed{\begin{tikzcd}[row sep=small] C^0_{harm} & C^1_{harm} & C^2_{harm} & & C^{n-1}_{harm} & C^n_{harm} \\ \oplus & \oplus & \oplus & & \oplus & \oplus \\ & C^1_{ex} & C^2_{ex} & \cdots & C^{n-1}_{ex} & C^n_{ex} \arrow[dl, dashed, "K"] \\ C^0_{coex} \arrow[uur, "d"] & C^1_{coex} \arrow[u, phantom, "\oplus"] \arrow[uur, "d"] \arrow[ul, dashed] & C^2_{coex} \arrow[ul, dashed, "K"] & & C^{n-1}_{coex} \arrow[ur, "d"] & \end{tikzcd}}$$

$$\tilde{\tau} = \prod_{j=0}^{n-1} \left(\det_{C^j_{coex} \to C^{j+1}_{ex}}(d)\right)^{(-1)^j} = \prod_{j=0}^{n} \left(\det{}'_{C^j} \Delta\right)^{-\frac{(-1)^j j}{2}} \in \mathbb{R}_{>0}$$

<u>determinant lines</u> $\mathrm{Det}\, C^\bullet = \bigotimes_{j=0}^{n} (\Lambda^{top} C^j)^{(-1)^j}$

notation: $L^{-1} := L^*$

differential d induces an isomorphism of determinant lines

$$\boxed{\mathbb{T}: \mathrm{Det}\, C^\bullet \xrightarrow{\simeq} \mathrm{Det}\, H^\bullet}$$

- from $0 \to C^j_{cl} \to C^j \xrightarrow{d} C^{j+1}_{ex} \to 0$

$\Rightarrow \mathrm{Det}\, C^\bullet \simeq \underbrace{\mathrm{Det}\, C^\bullet_{cl}}\otimes (\mathrm{Det}\, C^\bullet_{ex})^{-1}$

$0 \to C^j_{ex} \to C^j_{cl} \to H^j \to 0$

$\Rightarrow \mathrm{Det}\, C^\bullet_{cl} \simeq \mathrm{Det}\, C^\bullet_{ex} \otimes \mathrm{Det}\, H^\bullet$

inner product $(,)$ induces a ~~canonical~~ preferred element $\mu \in \mathrm{Det}\, C^\bullet$ - by wedging any o/n basis in $C^\bullet$

and an element $\mu_H \in \mathrm{Det}\, H^\bullet$

then the torsion is: $\tau(C^\bullet, d, (,)) = \mathbb{T}(\mu) = \tilde{\tau} \cdot \mu_H \in \mathrm{Det}\, H^\bullet / \{\pm 1\}$

<u>Integral formula:</u> view $C^\bullet$ as a $\mathbb{Z}$-graded vector space

$\mathcal{F} := C^\bullet[1] \oplus (C^\bullet)^*[-2]$ - "space of (BV) fields"

		C^0	C^1	C^2	C^3	...	C^n
$(C^n)^*$	...	$(C^3)^*$	$(C^2)^*$	$(C^1)^*$	$(C^0)^*$		
$\mathcal{F}^{2-n}$	...	$\mathcal{F}^{-1}$	$\mathcal{F}^0$	$\mathcal{F}^1$	$\mathcal{F}^2$	...	$\mathcal{F}^{n-1}$

fields: $(A, B) \in \mathcal{F}$ — A: cochain, B: chain

$A = A^{(0)} + \dots + A^{(n)}$ ← cochain degree

$B = B^{(0)} + \dots + B^{(n)}$ ← chain degree

$A^{(j)}$ is $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$ for $j \begin{pmatrix}\text{odd}\\ \text{even}\end{pmatrix}$ $\quad B^{(j)}$ is $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$ for $j \begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$

- $\mathcal{F}$ has a canonical -1-symplectic structure $\omega: \mathcal{F}^j \otimes \mathcal{F}^{1-j} \to \mathbb{R}$

let $(C^\bullet, d)$ be acyclic, K - a chain contraction

$\mathcal{L} = C^\bullet_{coex}[1] \oplus (C^\bullet_{coex})^*[-2] \quad \subset \mathcal{F}$ a Lagrangian subspace - "gauge-fixing"

$S = \langle B, dA \rangle = \sum_{j=0}^{n-1} \langle B^{(j+1)}, dA^{(j)} \rangle$ - "action functional"

- $\mathrm{Dens}^{1/2}\, \mathcal{F} \cong \mathrm{Dens}\, \underbrace{\lambda}_{\text{any Lagrangian } \lambda \subset \mathcal{F}}$. Thus: $\mathrm{Dens}^{1/2}\, \mathcal{F} \cong \mathrm{Dens}\, C^\bullet[1] \cong \mathrm{Det}\, C^\bullet /_{\pm 1} \ni \mu$ ← preferred element

<u>Theorem:</u> for $C^\bullet$ acyclic,

$$\int_{\mathcal{L} \subset \mathcal{F}} e^{\frac{i}{\hbar} S(A,B)} \mu^\hbar = \tau(C^\bullet, d, (,))$$

where $\mu^\hbar = \xi^\hbar_S \cdot \mu$ and

$$\xi^\hbar = \prod_{j=0}^{n} \Big(\underbrace{(2\pi\hbar)^{-\frac{1}{4} + \frac{1}{2} j (-1)^{j-1}} (e^{-\frac{\pi i}{2}} \hbar)^{\frac{1}{4} + \frac{1}{2} j (-1)^{j-1}}}_{\xi^\hbar_j} \Big)^{\dim C^j}$$

Torsion 3

Model Gaussian (Fresnel) integrals:

pair of even variables: $\displaystyle\int_{V\oplus V^*} da\, db\, e^{\frac{i}{\hbar}\langle b, Ma\rangle} = (2\pi\hbar)^{\dim V} (\det M)^{-1}$ $\qquad M \in \mathrm{End}\, V$

pair of odd variables: $\displaystyle\int_{\Pi V\oplus \Pi V^*} D\alpha\, D\beta\, e^{\frac{i}{\hbar}\langle \beta, M\alpha\rangle} = \left(\frac{i}{\hbar}\right)^{\dim V} \cdot \det M$

non-acyclic case: choose an (i,p,K)-triple $\quad C^\bullet \overset{(i,p,K)}{\rightsquigarrow} H^\bullet$

Hodge decomp. $C^\bullet = \underbrace{C^\bullet_{harm}}_{\simeq H^\bullet} \oplus C^\bullet_{ex} \oplus C^\bullet_{coex}$ induces symplectic splitting

$\mathcal{F} = \underbrace{\mathcal{F}'}_{H^\bullet[1]\oplus (H^\bullet)^*[-2]} \oplus \mathcal{F}''$ $\leftarrow$ $\ker p\,[1] \oplus (\ker p)^*[-2]$ $\quad$ and a Lagrangian $\mathcal{L} \subset \mathcal{F}''$

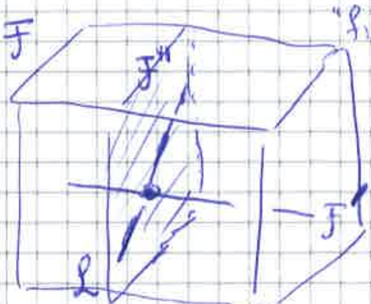
"slow fields" $\qquad$ "fast fields"

Theorem':

$$\int_{\mathcal{L}\subset\mathcal{F}''} e^{\frac{i}{\hbar}S(A,B)}\,\mu^\hbar = \xi^\hbar_{H^\bullet}\cdot \tau(C^\bullet, d, (,)) \quad \in \mathbb{C}\otimes \mathrm{Det}\, H^\bullet/\{\pm 1\}$$

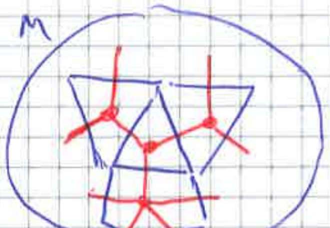
with $\xi^\hbar_{H^\bullet} = \prod_{j=0}^{n} (\xi^\hbar_j)^{\dim H^j} \in \mathbb{C}$

"fiber BV integral"



Geometrical setting (cellular complexes)

M - a closed manifold, X - CW decomposition $X^\vee$ - dual CW decomposition



E - $O(m)$-local system on M

$C^\bullet = C^\bullet(X,E)$ - cellular cochains

$\mathcal{F}_X = C^\bullet(X,E)[1] \oplus C_{-\bullet}(X,E^*)[-2]$ $\swarrow$ model I

or

$C^\bullet(X,E)[1] \oplus C^\bullet(X^\vee, E^*)[n-2]$ $\leftarrow$ model II (uses Poincaré duality)

fields $\quad A = \sum_{e\subset X} e^* \underset{\mathbb{R}^m}{A_e}$, $gh = 1-|e|$ $\qquad B = \sum_{e^\vee \subset X^\vee} (e^\vee)^* \underset{(\mathbb{R}^m)^*}{B_{e^\vee}}$, $gh = n-2-|e^\vee|$

$S_X = \langle B, dA\rangle_X = \sum_{\substack{e,e'\subset X\\ e'\subset\partial e}} B_{e^\vee} A_{e'}$ a boundary cell $\qquad \mu_X \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F} \cong \mathrm{Det}\, C^\bullet(X,E)$ - standard cellular half-density.

Partition function: from $C^\bullet(X,E) \overset{(i,p,K)}{\rightsquigarrow} H^\bullet(M,E)$, one infers the splitting $\mathcal{F}_X = \mathcal{F}' \oplus \mathcal{F}''$ and a Lagrangian $\mathcal{L} \subset \mathcal{F}''$

$$Z(X,E) = \int_{\mathcal{L}\subset\mathcal{F}''} e^{\frac{i}{\hbar} S_X(A,B)} \mu_X^\hbar = \xi^\hbar_{H^\bullet} \cdot \underbrace{\tau(X,E)}_{\text{Reidemeister torsion}} \in \mathbb{C} \otimes \mathrm{Det}\, H^\bullet(M,E)/_{\{\pm 1\}}$$

– does not depend on X! (only on M).

(Normalized) two-point correlation function:

$$\langle\langle A_{e'} B_{e^\vee} \rangle\rangle = \frac{1}{Z} \int A_{e'} B_{e^\vee} e^{\frac{i}{\hbar} S_X} \mu_X^\hbar = K_{e'e} \in E_{e'} \otimes E_e^*$$

– matrix element of $K: C^\bullet(X,E) \to C^{\bullet-1}(X,E)$

– "Green's function"

satisfies a difference equation corresp. to $dK + Kd = id - ip$

Field theory setting (on manifolds)

A.S. Schwarz '78: M - closed n-mfld, E - acyclic local system

abelian BF theory

$$\int_{\Omega^1(M,E)\oplus\Omega^{n-2}(M,E^*)} e^{\frac{i}{\hbar}\int_M \langle B, d_E A\rangle} DA\,DB \rightsquigarrow \begin{bmatrix}\text{gauge}\\ \text{-fixing}\end{bmatrix} \rightsquigarrow \int e^{\frac{i}{\hbar}\int_M \langle \mathcal{B}, d_E \mathcal{A}\rangle} D\mathcal{A}\, D\mathcal{B} \rightsquigarrow$$

$(\mathcal{A},\mathcal{B}) \in \Omega^\bullet(M,E)[1] \oplus \Omega^\bullet(M,E^*)[n-2]$ such that $\left.\begin{matrix} d_E^* \mathcal{A} = 0 \\ d_E^* \mathcal{B} = 0 \end{matrix}\right\}$ "Lorentz gauge"

∞-dimensional Gaussian integral; ζ-regularization

$$\rightsquigarrow Z = \prod_{j=0}^n \left(\det{}^{\zeta\text{-reg}}_{\Omega^j} \Delta_E\right)^{-\frac{(-1)^j \cdot j}{2}}$$

$=$ Ray-Singer torsion of (M,E).

Witten '89: M - closed oriented 3-mfld, $G = SU(2)$ (can be generalized to other groups)

fields: connections (in the trivial G-bundle) on M, $A \in \mathrm{Conn}_M \simeq \Omega^1(M,\mathfrak{g})$

action: $S_{CS}(A) = \int_M \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A\wedge A\wedge A\right)$

$$Z = \int_{\mathrm{Conn}_M} DA\, e^{\frac{i}{\hbar} S_{CS}(A)}$$

by stationary phase: crit. points of S_{CS} are <u>flat connections</u>

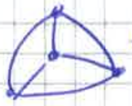
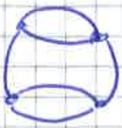
contribution of an <u>acyclic</u> flat connection A_0:

$$Z^{pert}_{A_0} = \int_{\Omega^1(M,\mathfrak{g})} Da\, e^{\frac{i}{\hbar} \overbrace{S_{CS}(A_0+a)}^{S_{CS}(A_0) + \int_M \mathrm{tr}\left(\frac{1}{2} a\, d_{A_0} a + \frac{1}{3} a\wedge a\wedge a\right)}}$$

Witten / Axelrod-Singer

$$Z^{pert}_{A_0} \simeq \int_{\Omega^1(M,\mathfrak{g})} Da\; e^{\frac{i}{\hbar} S_{CS}(A_0+a)} \xrightarrow[\text{fixing}]{\text{gauge}} \int_{\mathcal{L}\subset\Omega^1(M,\mathfrak{g})[\varepsilon]} D\tilde{a}\; e^{\frac{i}{\hbar} S_{CS}(A_0+\tilde{a})}$$

given by $d^*_{A_0}\tilde{a}=0$ – Lorentz gauge

$$= e^{\frac{i}{\hbar}S_{CS}(A_0)} \cdot \tau(M,A_0)^{\frac{1}{2}}\, e^{\frac{\pi i}{4}\eta(A_0,g)} \cdot e^{\frac{i}{24}\frac{\dim G}{2\pi} S_{grav}(g,\varphi)} \cdot \Big(1 + \;$$  $$\; + \;$$  $$\; + \ldots\Big)$$

$+$  $+$ ... $\propto \hbar$ $\qquad$ $\propto \hbar^2$

here: $\tau(M,A_0)$ – Ray-Singer torsion

$\eta(A_0,g) = \lim_{s\to 0} \sum \mathrm{sign}(\lambda)\, \lambda^{-s}$ – APS η-invariant of the operator $L_- = *d_{A_0} + d_{A_0}*$ $\mathrel{\circlearrowright}\Omega^{odd}(M,\mathfrak{g})$

S_{grav} – Chern-Simons action for Levi-Civita connection

φ – framing of M

> cf. Fresnel integral $\displaystyle\int_V e^{\frac{i}{2}(a,Pa)}\, da = (2\pi)^{\frac{\dim V}{2}} \cdot (\det P)^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}\mathrm{sign}\, P}$

- ~~result~~ $Z^{pert}_{A_0}$ independent of g (but depends on φ)
- corrections are given by configuration space integrals; integrand – product of Green's functions ~~integral~~ $\varkappa \in \Omega^2(\mathrm{Conf}_2(M), E\boxtimes E^*)$ – integral kernel for $K = d^*_{A_0}\Delta^{-1}_{A_0}$ – Hodge-de Rham chain contraction

---

Non-abelian BF theory on a closed surface Σ of genus $g\geq 2$ (Witten, 1991)

$$\int_{\Omega^1(\Sigma,\mathfrak{g})\oplus\Omega^0(\Sigma,\mathfrak{g}^*)} e^{\frac{i}{\hbar}\int_\Sigma \mathrm{tr}\, B\wedge(dA+A\wedge A)} \rightsquigarrow \int_{\mathcal{M}^{irred}_\Sigma} \tau(\Sigma, A_0) =: Z$$

gauge-fixing, crit. points are ($A=A_0$ – flat connections) $B=0$

τ – R-torsion

$\mathcal{M}^{irred}_\Sigma$ – moduli space of (irreducible) flat connections $\subset \mathrm{Hom}(\pi_1(\Sigma), G)/G$

$$\tau(\Sigma,A_0) \in \mathrm{Det}\, H^\bullet_{d_{A_0}} = \left(\Lambda^{top} H^1_{d_{A_0}}\right)^* \cong \Lambda^{top} T^*_{[A_0]}\mathcal{M}_\Sigma$$

$\Rightarrow \tau(\Sigma,A_0)$ is a volume form on $\mathcal{M}_\Sigma$. In fact (computation),

$\tau(\Sigma,A_0) = \frac{\omega^N}{N!}$ – Liouville volume form for Atiyah-Bott symp. structure on $\mathcal{M}_\Sigma$,

$N = \frac{1}{2}\dim\mathcal{M}_\Sigma = (g-1)\dim G$.

So: Z = symplectic volume of $\mathcal{M}^{irred}_\Sigma$ $= \#Z(G)\,(\mathrm{Vol}\, G)^{2g-2} \sum_{R\text{ – irrep of } G} \frac{1}{(\dim R)^{2g-2}}$

for $G = SU(2)$, $Z \propto B_{2g-2}$ Bernoulli number