2.5 Accelerating Convergence

Example. The Black-Scholes formula – A problem has "complicated" derivative

The Black-Scholes formula for a European call option is given by: $C = S_0 N(d_1) - Ke^{-rt} N(d_2)$.

C is the call price, S_0 is the price of the underlying asset at $t = 0$, K is the strike price at the maturity, r is the risk-free interest rate, $N(d)$ is the cumulative distribution function of the standard normal probability distribution, $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$, and $d_2 = d_1 - \sigma\sqrt{t}$. σ is the variability in the marked price known as the volatility.

Q: Given a target price C^*, what is the corresponding volatility σ_*?

Solution: Find the root of $f(\sigma) = S_0 N(d_1) - Ke^{-rt} N(d_2) - C^*$.

$$\sigma_{n+1} = \sigma_n - \alpha f(\sigma_n)$$

Where α is a small value.

Aitken's Δ^2 Method

- **Assume** $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p.
- Further assume $\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p}$ when n is large
- Solving for p yields:

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

A little algebraic manipulation gives:

$$p \approx p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

- **Define** $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

Remark: The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster.

Definition 2.13

The **forward difference** Δp_n is defined by

$\Delta p_n = p_{n+1} - p_n$. High powers of Δ are defined recursively by $\Delta^k p_n = \Delta\left(\Delta^{k-1} p_n\right)$.

Remark: $\widehat{p_n}$ can also be rewritten as

$$\widehat{p_n} = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

Theorem 2.14:

Suppose that $\{p_n\}_{n=0}^{\infty}$ converges linearly to the limit p and that $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} < 1$. Then the Aitken's Δ^2 sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that $\lim_{n\to\infty} \frac{\widehat{p_n}-p}{p_n-p} = 0$.

Example. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ generated by the fixed point iteration $p_{n+1} = \cos(p_n)\,, p_0 = 0.$

iteration	p_n	$\widehat{p_n}$
0	0.00000000000000	0.68507335732604 5
1	1.00000000000000	0.7 2801036146761 7
2	0.54030230586814 0	0.73 366516458523 1
3	0.85755321584639 3	0.73 690629434047 4
4	0.65428979049777 9	0.73 805042137166 4
5	0.7 9348035874256 6	0.73 863609688165 5
6	0.7 0136877362275 7	0.73 887658281713 6
7	0.7 6395968290065 4	0.73 899224302703 4
8	0.7 2210242502670 8	0.7390 4251132815 9
9	0.7 5041776176376 1	0.7390 6594959994 1
10	0.73 1404042422510	0.7390 7638331895 6
11	0.7 4423735490055 7	0.73908 1177259563*
12	0.73 5604740436347	0.73908 3333909684*

Steffensen's Method

- Steffensen's Method combines fixed-point iteration and the Aitken's Δ^2 method:

Step 0. Suppose we have a fixed point iteration:

$$p_0, \qquad \mathrm{p}_1 = \mathrm{g}(\mathrm{p}_0), \qquad \mathrm{p}_2 = \mathrm{g}(\mathrm{p}_1)$$

Once we have we have p_0, p_1 and p_2, we can compute

$$p_0^{(1)} = p_0 - \frac{(p_1 - p_0)^2}{(p_2 - 2p_1 + p_0)}$$

Step 1. Then we "restart" the fixed point iteration with

$$p_1^{(1)} = g\left(p_0^{(1)}\right), \quad p_2^{(1)} = g\left(p_1^{(1)}\right)$$

and compute:

$$p_0^{(2)} = p_0^{(1)} - \frac{\left(p_1^{(1)} - p_0^{(1)}\right)^2}{\left(p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}\right)}.$$

Step 2. We "restart" the fixed point iteration with

$$p_1^{(2)} = g\left(p_0^{(2)}\right), \quad p_2^{(2)} = g\left(p_1^{(2)}\right)$$

and compute:

$$p_0^{(3)} = p_0^{(2)} - \frac{\left(p_1^{(2)} - p_0^{(2)}\right)^2}{\left(p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}\right)}.$$

Example. Compare fixed-point iteration, Newton's method and Steffensen's method for solving:

$$f(x) = x^3 + 4x^2 - 10 = 0.$$

Solution: $$x^3 + 4x^2 = 10$$

$$x^2(x+4) = 10$$

$$x^2 = \frac{10}{x+4}$$

Fixed point iteration: $p_{n+1} = g(p_n) = \sqrt{\frac{10}{p_n+4}}$

i	p_n	$g(p_n)$
0	1.50000	1.34840
1	1.34840	1.36738
2	1.36738	1.36496
3	1.36496	1.3652
4	1.36526	1.36523
5	1.36523	1.36523

2. Newton's method

i	x_n	$f(x_n)$
0	1.50000	1.51600e-01
1	1.36495	-3.11226e-04
2	1.36523	-1.35587e-09

3. Steffensen's method

$p_0^{(0)}$	$p_1^{(0)}$	$p_2^{(0)}$	$p_0^{(1)} = \{\Delta^2\}(p_0^{(0)})$	$\lvert p_2^{(0)} - p_0^{(1)}\rvert$
1.50000	1.34840	1.36738	1.36527	3.96903e-05
	$p_1^{(1)}$	$p_2^{(1)}$	$p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$	$\lvert p_2^{(1)} - p_0^{(2)}\rvert$
	1.36523	1.36523	1.36523	2.80531e-12