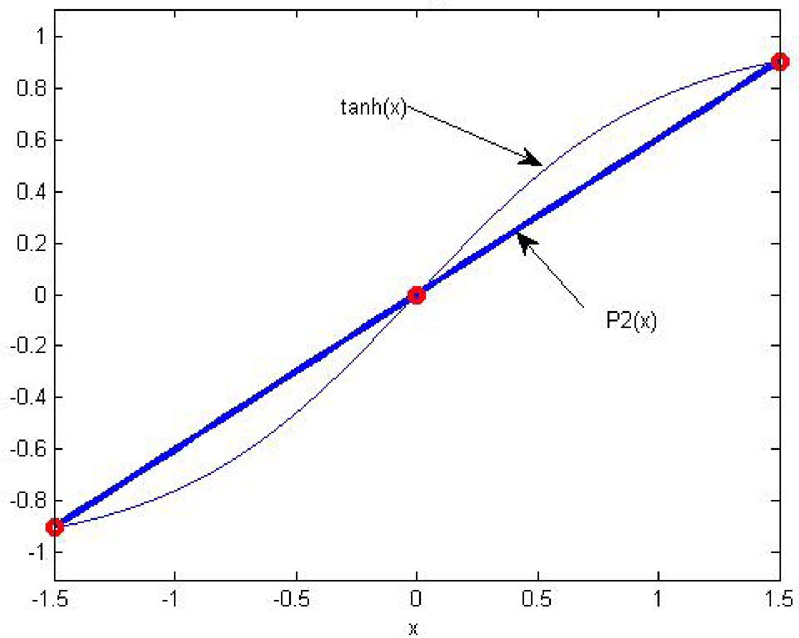
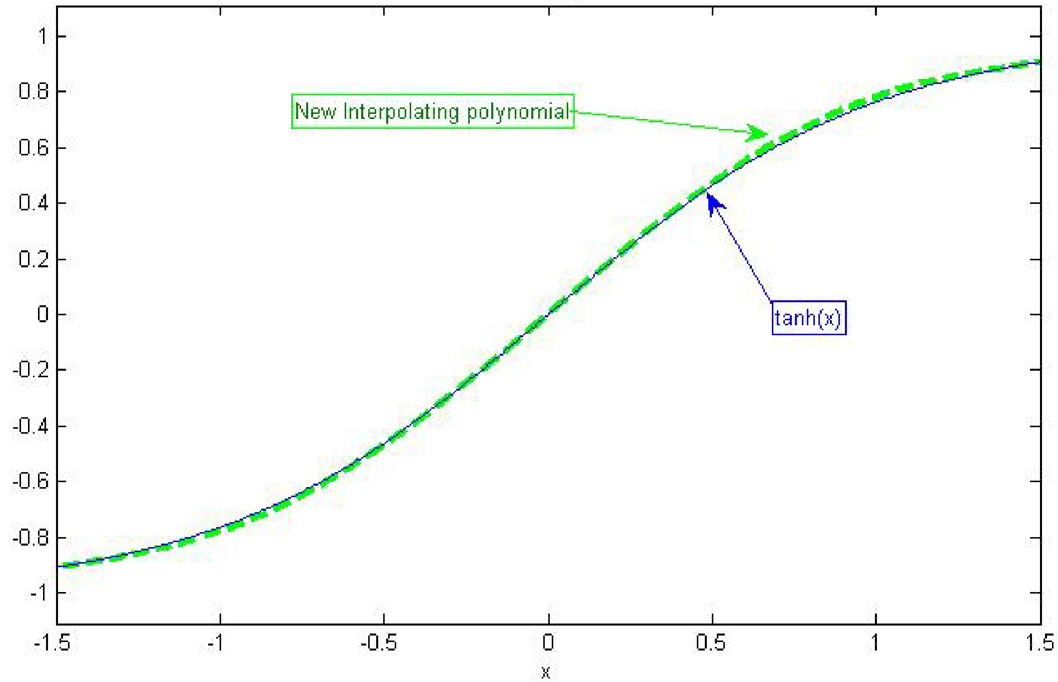
3.4 Hermite Interpolation
3.5 Cubic Spline Interpolation

Illustration. Consider to interpolate $\tanh(x)$ using Lagrange polynomial and nodes $x_0 = -1.5, x_1 = 0, x_2 = 1.5$.



Now interpolate $\tanh(x)$ using nodes $x_0 = -1.5, x_1 = 0, x_2 = 1.5$. Moreover, Let 1^{st} derivative of interpolating polynomial agree with derivative of $\tanh(x)$ at these nodes.



Remark:This is called Hermite interpolating polynomial.

Hermite Polynomial

Definition. Suppose $f \in C^1[a,b]$. Let $x_0, \ldots, x_n$ be distinct numbers in $[a,b]$, the Hermite polynomial $P(x)$ approximating f is that:

1. $P(x_i) = f(x_i)$, for $i = 0, \ldots, n$

2. $\frac{dP(x_i)}{dx} = \frac{df(x_i)}{dx}$, for $i = 0, \ldots, n$

Remark: $P(x)$ and $f(x)$ agree not only function values but also 1st derivative values at x_i, $i = 0, \ldots, n$.

Osculating Polynomials

Definition 3.8 Let $x_0, \dots, x_n$ be distinct numbers in $[a, b]$ and for $i = 0, \dots, n$, let m_i be a nonnegative integer. Suppose that $f \in C^m[a, b]$, where $m = \max_{0 \le i \le n} m_i$. The osculating polynomial approximating f is the polynomial $P(x)$ of least degree such that $\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$ for each $i = 0, \dots, n$ and k$= 0, \dots, m_i$.

Remark: the degree of $P(x)$ is at most $M = \sum_{i=0}^{n} m_i + n$.

Theorem 3.9 If $f \in C^1[a,b]$ and $x_0, \dots, x_n \in [a,b]$ distinct numbers, the Hermite polynomial of degree at most $2n+1$ is:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

Where

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L_{n,j}^2(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j) L_{n,j}^2(x)$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \dots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some $\xi(x) \in (a,b)$.

Remark:

1. $H_{2n+1}(x)$ is a polynomial of degree at most $2n+1$.
2. $L_{n,j}(x)$ is jth Lagrange basis polynomial of degree n.
3. $\frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$ is the error term.

Remark:

1. When $i \neq j$: $H_{n,j}(x_i) = 0; \widehat{H}_{n,j}(x_i) = 0.$
2. When $i = j$:

$$\begin{cases} H_{n,j}(x_j) = [1 - 2(x_j - x_j)L'_{n,j}(x_j)]L_{n,j}^2(x_j) = 1 \\ \widehat{H}_{n,j}(x_j) = (x_j - x_j)L_{n,j}^2(x_j) = 0 \end{cases}$$

$\Longrightarrow H_{2n+1}(x_j) = f(x_j).$

3. $H'_{n,j}(x) = L_{n,j}(x)\left[-2L'_{n,j}(x_j)L_{n,j}(x) + \left(1 - 2(x - x_j)L'_{n,j}(x_j)\right) 2L'_{n,j}(x)\right]$

$\Longrightarrow$ When $i \neq j$: $H'_{n,j}(x_i) = 0$; When $i = j$: $H'_{n,j}(x_j) = 0.$

4. $\widehat{H}'_{n,j}(x) = L_{n,j}^2(x) + 2(x - x_j)L_{n,j}(x)L'_{n,j}(x)$

$\Longrightarrow$ When $i \neq j$: $\widehat{H}'_{n,j}(x_i) = 0$; When $i = j$: $\widehat{H}'_{n,j}(x_j) = 1.$

Example 3.4.1 Use Hermite polynomial that agrees with the data in the table to find an approximation of $f(1.5)$

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

3rd Degree Hermite Polynomial

- Given distinct x_0, x_1 and values of f and f' at these numbers.

$$H_3(x) = \left(1 + 2\,\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f(x_0) + (x - x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f'(x_0) + \left(1 + 2\,\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f(x_1) + (x - x_1)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f'(x_1)$$

Hermite Polynomial by Divided Differences

Suppose $x_0, \dots, x_n$ and f, f' are given at these numbers. Define $z_0, \dots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \qquad \text{for } i = 0, \dots, n$$

Construct divided difference table, but use

$$f'(x_0), f'(x_1), .., f'(x_n)$$

to set the following undefined divided difference:

$$f[z_0, z_1], f[z_2, z_3], \dots, f[z_{2n}, z_{2n+1}].$$

Namely, $f[z_0, z_1] = f'(x_0)$, $f[z_2, z_3] = f'(x_1)$, ...

$$f[z_{2n}, z_{2n+1}] = f'(x_n).$$

The Hermite polynomial is

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0) \dots (x - z_{k-1})$$

Example 3.4.2 Use divided difference method to determine the Hermite polynomial that agrees with the data in the table to find an approximation of $f(1.5)$

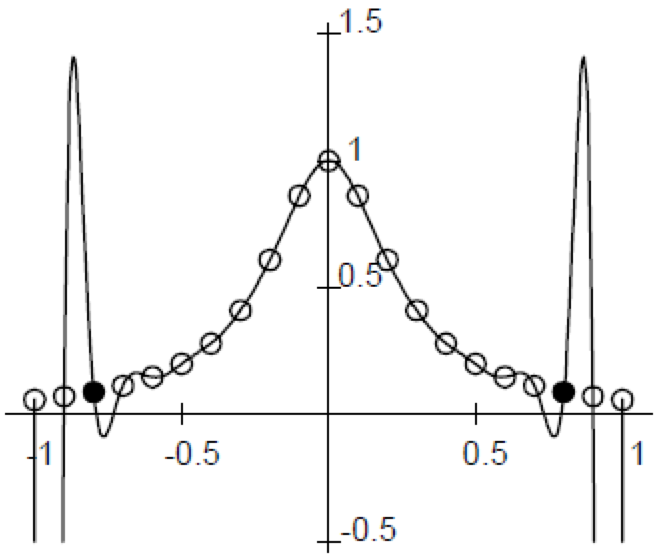
k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

Divided Difference Notation for Hermite Interpolation

- Divided difference notation for Hermite polynomial interpolating 2 nodes: x_0, x_1.

$$H_3(x)$$
$$= f(x_0) + f'(x_0)(x - x_0) + f[x_0, x_0, x_1](x - x_0)^2$$
$$+ f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1)$$

Problems with High Order Polynomial Interpolation

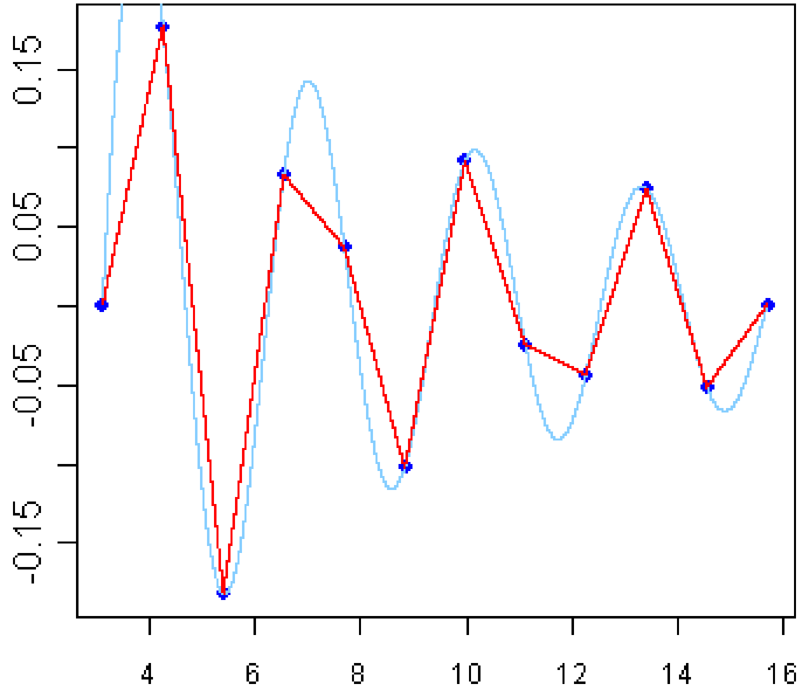


- 21 equal-spaced numbers to interpolate $f(x) = \frac{1}{1+25x^2}$. The interpolating polynomial oscillates between interpolation points.

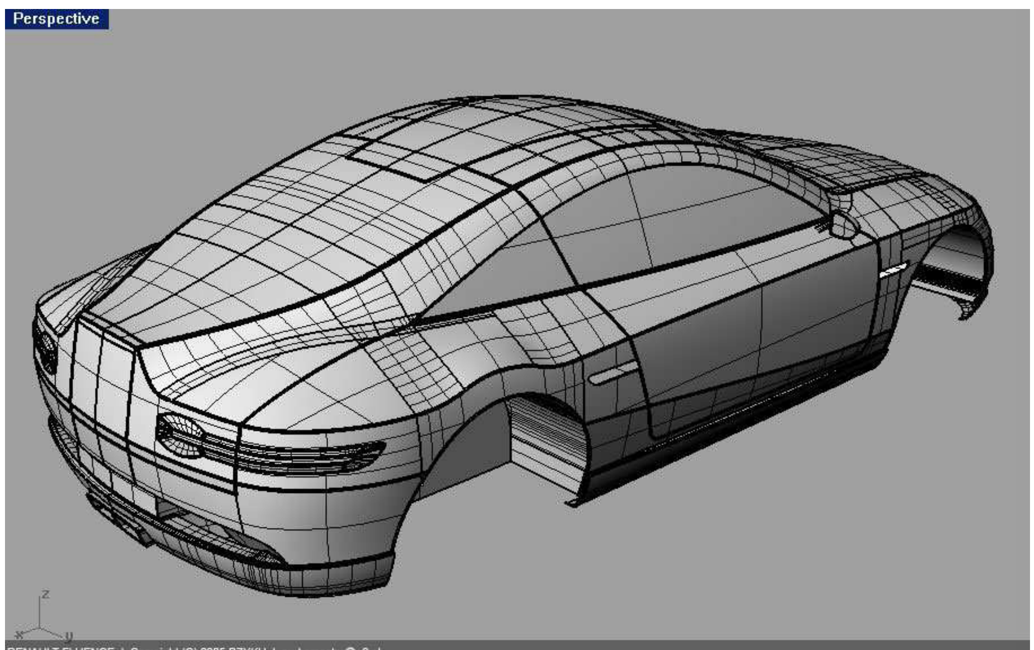
3.5 Cubic Splines

- Idea: Use piecewise polynomial interpolation, i.e, divide the interval into smaller sub-intervals, and construct different low degree polynomial approximations (with small oscillations) on the sub-intervals.

Example. *Piecewise-linear* interpolation



- **Challenge**: If $f'(x_i)$ are not known, can we still generate interpolating polynomial with continuous derivatives?
- **Spline**: A spline consists of a long strip of wood (a lath) fixed in position at a number of points. The lath will take the shape which minimizes the energy required for bending it between the fixed points, and thus adopt the smoothest possible shape.



- In mathematics, a spline is a function that is piecewise-defined by polynomial functions, and which possesses a high degree of smoothness at the places where the polynomial pieces connect.
- **Example**. Irwin-Hall distribution. Nodes are -2, -1, 0, 1, 2.

$$f(x) = \begin{cases} \frac{1}{4}(x+2)^3 & -2 \le x \le -1 \\ \frac{1}{4}(3|x|^3 - 6x^2 + 4) & -1 \le x \le 1 \\ \frac{1}{4}(2-x)^3 & 1 \le x \le 2 \end{cases}$$

Notice: $f'(-1) = \frac{3}{4},\ f'(1) = -\frac{3}{4}, f''(-1) = \frac{6}{4}, f''(1) = \frac{6}{4}$.

- Piecewise-polynomial approximation using cubic polynomials between each successive pair of nodes is called **cubic spline interpolation**.

Definition 3.10 Given a function f on $[a, b]$ and nodes $a = x_0 < \cdots < x_n = b$, a **cubic spline interpolant** S for f satisfies:

(a) $S(x)$ is a cubic polynomial $S_j(x)$ on $[x_j, x_{j+1}]$ with:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$
$$\forall j = 0,1, \dots, n - 1.$$

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1}), \forall j = 0,1, \dots, n - 1.$

(c) $S_j(x_{j+1}) = S_{j+1}(x_{j+1}), \forall j = 0,1, \dots, n - 2.$

Remark: (c) is derived from (b).

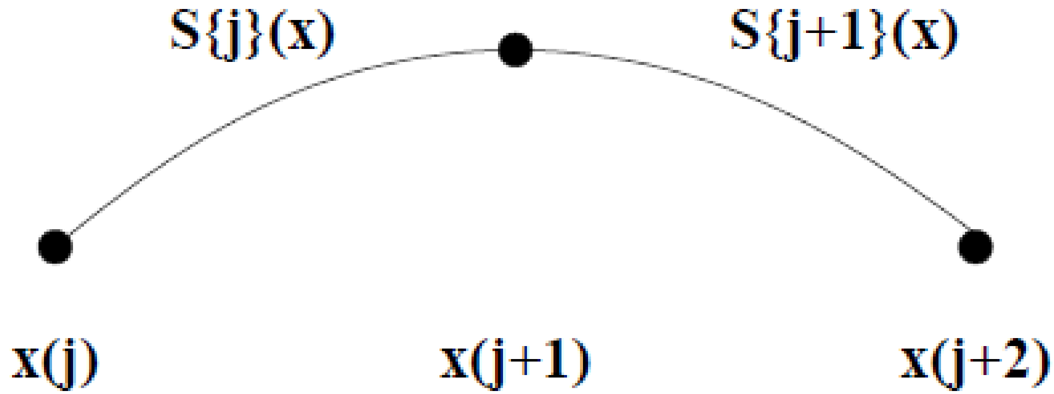
(d) $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}), \forall j = 0,1, \dots, n - 2.$

(e) $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}), \forall j = 0,1, \dots, n - 2.$

(f) One of the following boundary conditions:

(i) $S''(x_0) = S''(x_n) = 0$ (called free or natural boundary)

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (called clamped boundary)



The spline segment $S_j(x)$ is on $\left[x_j, x_{j+1}\right]$. The spline segment $S_{j+1}(x)$ is on $\left[x_{j+1}, x_{j+2}\right]$. Things to match at interior point x_{j+1}:

- Their function values: $S_j\left(x_{j+1}\right) = S_{j+1}\left(x_{j+1}\right) = f\left(x_{j+1}\right)$
- First derivative values: $S'_j\left(x_{j+1}\right) = S'_{j+1}\left(x_{j+1}\right)$
- Second derivative values: $S''_j\left(x_{j+1}\right) = S''_{j+1}\left(x_{j+1}\right)$

Example 3.5.1 Construct a natural spline $S(x)$ through (1,2), (2,3) and (3.5).

Building Cubic Splines

- Define: $S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$ and $h_j = x_{j+1} - x_j, \forall j = 0,1,\ldots,(n-1)$.
- Also define $a_n = f(x_n)$; $b_n = S'(x_n)$; $c_n = S''(x_n)/2$.

From Definition 3.10:

1) $S_j(x_j) = a_j = f(x_j)$ for $j = 0,1,\ldots,(n-1)$.
2) $S_{j+1}(x_{j+1}) = a_{j+1} = a_j + b_j h_j + c_j(h_j)^2 + d_j(h_j)^3$ for $j = 0,1,\ldots,(n-1)$.

   $Note: a_n = a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}(h_{n-1})^2 + d_{n-1}(h_{n-1})^3$
3) $S'_j(x_j) = b_j$, also $b_{j+1} = b_j + 2c_j h_j + 3d_j(h_j)^2$ for $j = 0,1,\ldots,(n-1)$.
4) $S''_j(x_j) = 2c_j$, also $c_{j+1} = c_j + 3d_j h_j$ for $j = 0,1,\ldots,(n-1)$.
5) Natural or clamped boundary conditions

Solve a_j, b_j, c_j, d_j by substitution:

1. Solve Eq. 4) for $d_j = \frac{c_{j+1} - c_j}{3h_j}$, and substitute into Eqs. 2) and 3) to get:

2. $$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}); \qquad (3.18)$$
$$b_{j+1} = b_j + h_j(c_j + c_{j+1}). \qquad (3.19)$$

3. Solve for b_j in Eq. (3.18) to get:
$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}). \qquad (3.20)$$
Reduce the index by 1 to get:
$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j).$$

4. Substitute b_j and b_{j-1} into Eq. (3.19):
$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \qquad (3.21)$$
$$\frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$
for $j = 1, 2, \ldots, (n-1)$.

Solving the Resulting Equations

$\forall j = 1,2,\ldots,(n-1)$

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) \qquad (3.21)$$

Remark: (n-1) equations for (n+1) unknowns $\{c_j\}_{j=0}^{n}$. Eq. (3.21) is solved with boundary conditions.

- Once compute c_j, we then compute:

$$b_j = \frac{(a_{j+1} - a_j)}{h_j} - \frac{h_j(2c_j + c_{j+1})}{3} \qquad (3.20)$$

and

$$d_j = \frac{(c_{j+1} - c_j)}{3h_j} \qquad (3.17) \text{ for } j = 0,1,2,\ldots,(n-1)$$

Building Natural Cubic Spline

- Natural boundary condition:
  1. $0 = S''_0(x_0) = 2c_0 \rightarrow c_0 = 0$
  2. $0 = S''_n(x_n) = 2c_n \rightarrow c_n = 0$

Step 1. Solve Eq. (3.21) together with $c_0 = 0$ and $c_n = 0$ to obtain $\{c_j\}_{j=0}^{n}$.

Step 2. Solve Eq. (3.20) to obtain $\{b_j\}_{j=0}^{n-1}$.

Step 3. Solve Eq. (3.17) to obtain $\{d_j\}_{j=0}^{n-1}$.

Building Clamped Cubic Spline

- Clamped boundary condition:

  a) $S'_0(x_0) = b_0 = f'(x_0)$

  b) $S'_{n-1}(x_n) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n) = f'(x_n)$

  Remark: a) and b) gives additional equations:

  $$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(x_0) \qquad (a)$$

  $$h_{n-1}c_{n-1} + 2h_{n-1}c_n = -\frac{3}{h_{n-1}}(a_n - a_{n-1}) + 3f'(x_n) \qquad (b)$$

Step 1. Solve Eq. (3.21) together with Eqs. (a) and (b) to obtain $\{c_j\}_{j=0}^{n}$.

Step 2. Solve Eq. (3.20) to obtain $\{b_j\}_{j=0}^{n-1}$.

Step 3. Solve Eq. (3.17) to obtain $\{d_j\}_{j=0}^{n-1}$.

Example 3.5.4 Let $(x_0, f(x_0)) = (0,1), (x_1, f(x_1)) = (1, e), (x_2, f(x_2)) = (2, e^2)$, $(x_3, f(x_3)) = (3, e^3)$. And $f'(x_0) = e$, $f'^{(x_3)} = e^3$. Determine the clamped spline $S(x)$.

Theorem 3.11 If f is defined at the nodes: $a = x_0 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes; that is a spline interpolant that satisfied the natural boundary conditions $S''(a) = 0, S''(b) = 0$.

Theorem 3.12 If f is defined at the nodes: $a = x_0 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes; that is a spline interpolant that satisfied the clamped boundary conditions $S'(a) = f'(a), S'(b) = f'(b)$.

Error Bound

Theorem 3.13 If $f \in C^4[a, b]$, let $M = \max_{a \le x \le b} |f^4(x)|$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes: $a = x_0 < \cdots < x_n = b$, then with

$$h = \max_{0 \le j \le n-1} \left(x_{j+1} - x_j\right)$$

$$\max_{a \le x \le b} |f(x) - S(x)| \le \frac{5Mh^4}{384}.$$