5.6 Multistep Methods

Motivation: Consider IVP: $y' = f(t, y), \quad a \le t \le b, \; y(a) = \alpha$. To compute solution at t_{i+1} , approximate solutions at mesh points $t_0, t_1, t_2, \ldots t_i$ are already obtained. Since in general error $|y(t_{i+1}) - w_{i+1}|$ grows with respect to time t, it then makes sense to use more previously computed approximate solution $w_i, \; w_{i-1}, w_{i-2}, \ldots$ when computing w_{i+1}.

Definition 5.14 An **m-step** multistep method for solving the IVP:

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \alpha$$

has a difference equation for computing w_{i+1} at the mesh point t_{i+1} represented by:

$$\begin{aligned} w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ +h[\boldsymbol{b_m} f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) \\ + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})] \end{aligned}$$

for $i = m-1, m, \ldots, N-1$, where $h = (b-a)/N$, the $a_0, \, a_1, \ldots, a_{m-1}$ and $b_0, \, b_1, \ldots, b_m$ are constants, and the starting values $w_0 = \alpha, w_1 = \alpha_1, \ldots, w_{m-1} = \alpha_{m-1}$ are specified.

Remark. 1. When $b_m = 0$, the method is called **explicit**;

2. When $b_m \neq 0$, the method is called **implicit**.

Adams-Bashforth two-step *explicit* method.

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{where } i = 1, 2, \ldots N-1.$$

Adams-Moulton two-step *implicit* method.

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})]$$

$$\text{where } i = 1, 2, \ldots N-1.$$

Adams-Bashforth four-step *explicit* method.

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \quad \text{where } i = 3, 4, \ldots N-1.$$

Adams-Moulton four-step *implicit* method.

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{720}[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 264f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2}) - 19f(t_{i-3}, w_{i-3})]$$

$$\text{where } i = 3, 4, \ldots N-1.$$

Example 1. Solve the IVP $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \ y(0) = 0.5$ by Adams-Bashforth four-step explicit method and Adams-Moulton two-step implicit method respectively. Use the Runge-Kutta method of order four to get needed starting values for approximation and $h = 0.2$.

Solution:

By using Runge-Kutta method of order four:

$$w_0 = 0.5$$
$$y(0.2) \approx w_1 = 0.8292933$$
$$y(0.4) \approx w_2 = 1.2140762$$
$$y(0.6) \approx w_3 = 1.6489220$$

Example. Derive Adams-Bashforth two-step *explicit* method for solving the IVP: $y' = f(t, y), \quad a \leq t \leq b, \; y(a) = \alpha$.

Integrate $y' = f(t, y)$ over $[y_i, y_{i+1}]$

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} y'(t)dt = \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

Then form interpolating polynomial through $\left(t_i, f\left(t_i, y(t_i)\right)\right)$, $\left(t_{i-1}, f\left(t_{i-1}, y(t_{i-1})\right)\right)$ to approximate $f\left(t, y(t)\right)$ and subsequently $\int_{t_i}^{t_{i+1}} f(t, y(t))dt$.

Definition 5.15 Local Truncation Error. If $y(t)$ solves the IVP $y' = f(t, y)$,
$a \le t \le b, \;\; y(a) = \alpha$ and

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

the local truncation error is:

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - a_{m-2}y(t_{i-1}) - \cdots - a_0 y(t_{i+1-m})}{h}$$
$$-[b_m f\big(t_{i+1}, y(t_{i+1})\big) + \cdots + b_0 f(t_{i+1-m}, y(t_i))]$$

for each $i = m-1, m, \ldots, N-1$.

NOTE: the local truncation error of a *m*-step *explicit* step is $O(h^m)$.

the local truncation error of a *m*-step *implicit* step is $O(h^{m+1})$.

Comparing *m*-step *explicit* step method *vs.* (*m-1)*-step *implicit* step method

a) both have the same order of local truncation error, $O(h^m)$.

b) Implicit method usually has greater stability and smaller round-off errors.

For example, local truncation error of Adams-Bashforth 3-step explicit method, $\tau_{i+1}(h) = \frac{3}{8} y^{(4)}(\mu_i) h^3$.

Local truncation error of Adams-Moulton 2-step implicit method, $\tau_{i+1}(h) = -\frac{1}{24} y^{(4)}(\xi_i) h^3$.

Predictor-Corrector Method

Motivation: **(1)** Solve the IVP $y' = e^y, \quad 0 \le t \le 0.25, \; y(0) = 1$ by the three-step Adams-Moulton method.

Solution: The three-step Adams-Moulton method is

$$w_{i+1} = w_i + \frac{h}{24}\left[9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}\right] \qquad Eq.(1)$$

$Eq.(1)$ can be solved by Newton's method. However, this can be quite computationally expensive.

(2) combine explicit and implicit methods.

4^{th} order Predictor-Corrector Method

(we will combine 4^{th} order Runge-Kutta method + 4^{th} order 4-step explicit Adams-Bashforth method + 4^{th} order 3-step implicit Adams-Moulton method)

Step 1: Use 4^{th} order Runge-Kutta method to compute w_0, w_1, w_2 and w_3.

Step 2: For $i = 4, 5, \ldots N$

(a) Predictor sub-step. Use 4^{th} order 4-step explicit Adams-Bashforth method to compute a predicated value $w_{i+1,p}$

$$w_{i+1,p} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

(b) Correction sub-step. Use 4th order three-step Adams-Moulton implicit method to compute a correction w_{i+1} (the approximation at $i+1$ time step)

$$w_{i+1} = w_i + \frac{h}{24}[9f\left(t_{i+1}, w_{i+1,p}\right) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]$$