6.1 Linear Systems of Equations

To solve a system of linear equations

$$E_1:\ \ a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$E_2:\ \ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$E_n:\ \ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n$$

for $x_1, x_2, \ldots, x_n$ by **Gaussian elimination with backward substitution**.

Matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Three elementary row operations.

1.Multiply one row by a nonzero number: $(\lambda E_i) \to (E_i)$

2.Interchange two rows: : $(E_j) \leftrightarrow (E_i)$

3.Add a multiple of one row to a different row: $\left(E_i + \lambda E_j\right) \to (E_i)$

Backward substitution

Example 1. To solve $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$

Solution: From $-\frac{5}{2}x_3 = -3$

$$x_3 = \frac{6}{5}.$$

Then from $2x_2 + x_3 = 4$

$$2x_2 + \frac{6}{5} = 4$$

$$x_2 = \frac{7}{5}.$$

Lastly from $x_1 + x_2 + 2x_3 = 6$

$$x_1 + \frac{7}{5} + 2\left(\frac{6}{5}\right) = 6$$

$$x_1 = \frac{11}{5}$$

Gaussian Elimination with Backward Substitution

1. Write the system of linear equations as an **augmented matrix** $[A \mid b]$.
2. Perform elementary row operations to put the augmented matrix in the upper triangular form
3. Solve the echelon form using backward substitution

Example 2. Solve the system of linear equations

$$\begin{aligned} 2x_2 + x_3 &= 4 \\ x_1 + x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + x_3 &= 7 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right] \underset{(E_1) \leftrightarrow (E_2)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

$$\underset{(E_3 - 2 * E_1) \rightarrow (E_3)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right] \underset{(E_3 + 0.5 * E_2) \rightarrow (E_3)}{\rightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -\frac{5}{2} & -3 \end{array}\right]$$

Now use backward substation to solve for values of x_1, x_2, x_3 (see **Example 1**).

Remark: Gaussian elimination is computationally expensive. The total number of multiplication and divisions is about $n^3/3$, where n is the number of unknowns.

6.2 Pivoting Strategies

Example 1. Apply Gaussian elimination to solve

$E1$: $0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding (The exact solution is $x_1 = 10.00$, $x_2 = 1.000$).

Remark. In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier $m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$ will be large, leading to large round-off errors.

To solve a system of linear equations

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for $x_1, x_2, \dots, x_n$ by **Gaussian elimination** where $a_{kk}^{(k)}$ are numbers with small magnitude.

Ideas of Partial Pivoting.

Partial pivoting finds the smallest $p \ \geq k$ such that

$$\left|a_{pk}^{(k)}\right| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$. $a_{pk}^{(k)}$ is used as the pivot element.

Example 2. Apply Gaussian elimination with partial pivoting to solve

$E1$: $0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

Step 1 of partial pivoting

$$\max\left\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\right\} = \{|0.003000|, |5.291|\} = 5.291 = |a_{21}^{(1)}|.$$

So perform $(E_1) \leftrightarrow (E_2)$ to make 5.291 the pivot element.

$$E1: \quad 5.291x_1 - 6.130x_2 = 46.78$$

$$E2: \quad 0.003000x_1 + 59.14x_2 = 59.17$$

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{0.003000}{5.29} = 0.0005670$$

Perform $\left(E_2 - m_{21}E_1\right) \rightarrow (E_2)$

$$5.291x_1 - 6.130x_2 = 46.78$$
$$59.14x_2 \approx 59.14$$

Backward substitution with 4-digit rounding: $x_1 = 10.00$; $x_2 = 1.000$.

Example 3. Apply Gaussian elimination with partial pivoting to solve

$E1$: $\quad 30.00x_1 + 591400x_2 = 591700$

$E2$: $\quad 5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{5.291}{30.00} = 0.1764$$

Perform $\left(E_2 - m_{21}E_1\right) \rightarrow (E_2)$:

$$30.00x_1 + 591400x_2 = 591700$$
$$-104300x_2 \approx -104400$$

Using backward substitution with 4-digit arithmetic:

$$x_1 = -10.00, \qquad x_2 = 1.001.$$

Scaled Partial Pivoting

- If there are large variations in magnitude of the elements within a row, scaled partial pivoting should be used.
- Define a scale factor s_i for each row

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

- At step i, find p (the element which will be used as pivot) such that

$$\frac{a_{pi}}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$$ and interchange the rows $(E_i) \leftrightarrow (E_p)$

NOTE: The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed.

Example 3. Apply Gaussian elimination with scaled partial pivoting to solve

$E1$: $30.00x_1 + 591400x_2 = 591700$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

$$s_1 = 591400$$

$$s_2 = 6.130$$

Consequently

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}$$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631$$

5.291 should be used as pivot element. So $(E_1) \leftrightarrow (E_2)$

Solve
$$\begin{aligned} 5.291x_1 - 6.130x_2 &= 46.78 \\ 30.00x_1 + 591400x_2 &= 591700 \end{aligned}$$

$x_1 = 10.00,\ x_2 = 1.000.$

Example 3. Use scaled partial pivoting with three-digit rounding to solve

$$\begin{aligned} 2.11x_1 - 4.21x_2 + 0.921x_3 &= 2.01 \\ 4.01x_1 + 10.2x_2 - 1.12x_3 &= -3.09 \\ 1.09x_1 + 0.987x_2 + 0.832x_3 &= 4.21 \end{aligned}$$