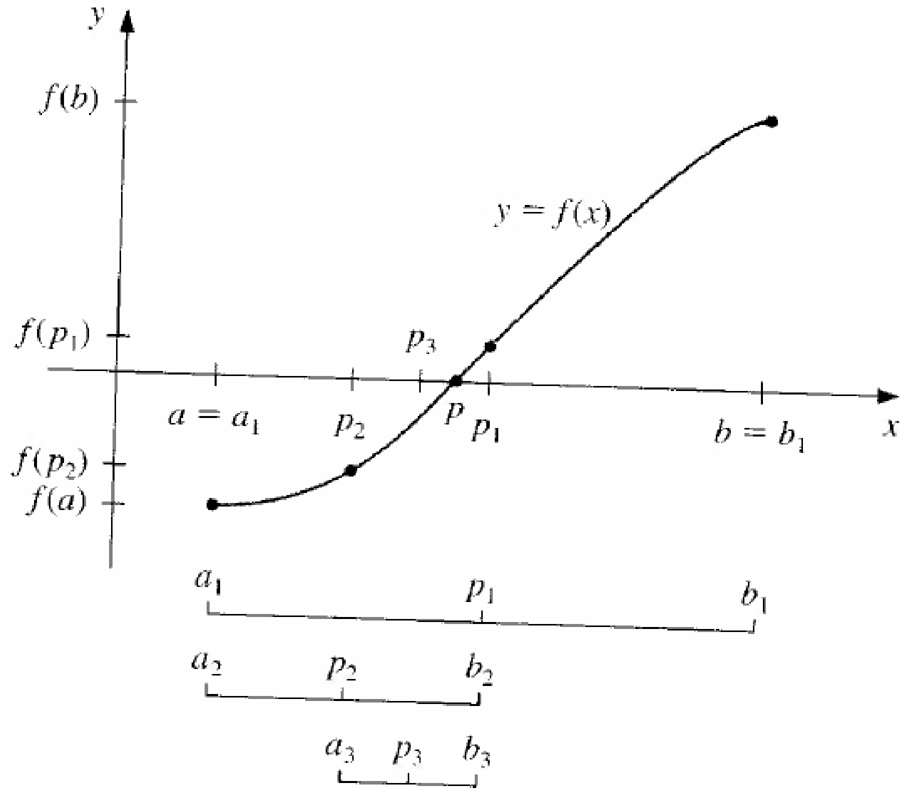
2.1 The Bisection Method

- **Intermediate Value Theorem**

If $f \in C[a,b]$, and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c) = K$.



- Assume $f(a_1)f(b_1) < 0$.
- Step one: compute $p_1 = \frac{a_1+b_1}{2}$. Test if $f(a_1)f(p_1) < 0$. If $f(a_1)f(p_1) < 0$, let $\mathrm{a}_2 = \mathrm{a}_1, \mathrm{b}_2 = p_1$. Otherwise, let $\mathrm{a}_2 = \mathrm{p}_1, \mathrm{b}_2 = b_1$.
- Step two: Compute $p_2 = \frac{a_2+b_2}{2}$. Test if $f(a_2)f(p_2) < 0$. If $f(a_2)f(p_2) < 0$, let $\mathrm{a}_3 = \mathrm{a}_2, \mathrm{b}_3 = p_2$. Otherwise, let $\mathrm{a}_3 = \mathrm{p}_2, \mathrm{b}_3 = b_2$.
- Repeat above step. p_n is approximate root.

Facts to remember:

1. The sequence of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ contains the desired root.

2. Intervals containing the root: $(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset (a_4, b_4) \ldots$

3. After n steps, the interval (a_n, b_n) has the length: $b_n - a_n = (1/2)^{n-1}(b - a)$

4. Let $p_n = \frac{b_n + a_n}{2}$ be the mid-point of (a_n, b_n). The limit of sequence $\{p_n\}_{n=1}^{\infty}$ is the root.

Convergence

- **Theorem 2.1**

Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of $f(x)$ with

$$|p_n - p| \leq \left({}^{1}/_{2}\right)^{n}(b-a), \qquad \text{when } n \geq 1$$

- **Convergence rate**

The sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with the rate of convergence $O\left(\left({}^{1}/_{2}\right)^{n}\right)$:

$$p_n = p + O\left(\left({}^{1}/_{2}\right)^{n}\right)$$

Example 2.1.1. Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in [1, 2], and use the Bisection method to determine an approximation to the root that has relative error within 10^{-4}.

Remark: $|p_n - p| \leq (1/2)^n (b - a)$

or $|p_n - p| \leq (1/2) \quad (b_n - a_n)$

- **Example 2.1.2.** Determine the number of iteration to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with absolute error smaller than 10^{-3}. Use $a_1 = 1, b_1 = 2$.

Solution: Since $|p_n - p| \leq (1/2)^n(b_1 - a_1) \leq 10^{-3}, \rightarrow$ $2^{-n}(2 - 1) \leq 10^{-3}$.

Solve for $n \rightarrow n \approx 9.96$.

So $n = 10$ is needed.

- **Exercise 2.1.13.** Find an approximation to $\sqrt[3]{25}$

Correct within 10^{-4} using bisection method.

Solution: Consider to solve $f(x) = x^3 - 25 = 0$ by the Bisection method.

By trial and error, we can choose $a_1 = 2, b_1 = 3$.

Because $f(a_1) \cdot f(b_1) < 0$.

The Algorithm

INPUT **a,b**; tolerance **TOL**; maximum number of iterations **N0**.

OUTPUT solution p or message of failure.

STEP1 Set i = 1;

FA = f(**a**);

STEP2 While i ≤ **N0** do STEPs 3-6.

STEP3 Set **p = a + (b-a)/2**; // a good way of computing middle point

FP = f(**p**).

STEP4 IF FP = 0 or (**b-a**) < TOL then

OUTPUT (**p**);

STOP.

STEP5 Set i = i +1.

STEP6 If FP·FA > 0 then

Set **a** = p;

FA = FP.

else

set **b** = p;

STEP7 OUTPUT(“Method failed after N0 iterations”);

STOP.