$$\varphi_m(\mathbf{x}_{Z(n,k)}) = \begin{cases} 1 & \text{if } m = n , \\ 0 & \text{if } m \neq n . \end{cases}$$

The right-hand side on K_k is computed by

$$f_m^k = \int_{K_k} f \varphi_m \, d\mathbf{x}, \quad m = 1, 2, 3 .$$

Note that m and n are the local numbers of the three vertices of K_k, while i and j used in (1.23) are the global numbers of vertices in K_h.

To assemble the global matrix $\mathbf{A} = (a_{ij})$ and the right-hand side $\mathbf{f} = (f_j)$, one loops over all triangles K_k and successively adds the contributions from different K_k's:

$$\begin{aligned} &\text{For } k = 1, 2, \ldots, \mathcal{M}, \text{ compute} \\ &a_{Z(m,k),Z(n,k)} = a_{Z(m,k),Z(n,k)} + a_{mn}^k , \\ &f_{Z(m,k)} = f_{Z(m,k)} + f_m^k, \quad m, \ n = 1, 2, 3 . \end{aligned}$$

The approach used is *element-oriented*; that is, we loop over elements (i.e., triangles). Experience shows that this approach is more efficient than the *node-oriented* approach (i.e., looping over all nodes); the latter approach wastes too much time in repeated computations of $\mathbf{A}$ and $\mathbf{f}$.

1.1.4.3 Solution of a Linear System

The solution of the linear system $\mathbf{Ap} = \mathbf{f}$ can be performed via a direct method (Gaussian elimination) or an iterative method (e.g., the conjugate gradient method), which will be discussed in Sect. 1.10. Here we just mention that in use of these two methods, it is not necessary to exploit an array $\mathbf{A}(M, M)$ to store the stiffness matrix $\mathbf{A}$. Instead, since $\mathbf{A}$ is sparse and usually a band matrix, only the nonzero entries of $\mathbf{A}$ need to be stored, say, in an one-dimensional array.

1.2 Sobolev Spaces

In the previous section, an introductory finite element method was developed for two simple model problems. To present the finite element method in a general formulation, we need to use function spaces. This section is devoted to the development of the function spaces that are slightly more general than the spaces of continuous functions with piecewise continuous derivatives utilized in the previous section. We establish the small fraction of these spaces that is sufficient to develop the foundation of the finite element method as studied in this book.

1.2.1 Lebesgue Spaces

In this section, we assume that Ω is an open subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, with piecewise smooth boundary. For a real-valued function v on Ω, we use the notation

$$\int_\Omega v(\mathbf{x})\, d\mathbf{x}$$

to denote the integral of f in the sense of Lebesgue (Rudin, 1987). For $1 \leq q < \infty$, define

$$\|v\|_{L^q(\Omega)} = \left(\int_\Omega |v(\mathbf{x})|^q\, d\mathbf{x}\right)^{1/q} .$$

For $q = \infty$, set

$$\|v\|_{L^\infty(\Omega)} = \text{ess } \sup\{|v(\mathbf{x})| : \mathbf{x} \in \Omega\} ,$$

where ess sup denotes the *essential supremum*. Now, for $1 \leq q \leq \infty$, we define the *Lebesgue spaces*

$$L^q(\Omega) = \{v :\ v \text{ is defined on } \Omega \text{ and } \|v\|_{L^q(\Omega)} < \infty\} .$$

For $q = 2$, for example, $L^2(\Omega)$ consists of all *square integrable functions* on Ω (in the sense of Lebesgue). To avoid trivial differences, we identify two functions u and v whenever $\|u - v\|_{L^q(\Omega)} = 0$; i.e., $u(\mathbf{x}) = v(\mathbf{x})$ for $\mathbf{x} \in \Omega$, except on a set of measure zero.

Given a linear (vector) space V, a *norm* in V, $\|\cdot\|$, is a function from V to $\mathbb{R}$ such that

- $\|v\| \geq 0 \quad \forall v \in V$; $\|v\| = 0$ if and only if $v = 0$.
- $\|cv\| = |c|\ \|v\| \quad \forall c \in \mathbb{R},\ v \in V$.
- $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$ (the triangle inequality).

A linear space V endowed with a norm $\|\cdot\|$ is called a *normed linear space*. V is termed *complete* if every *Cauchy sequence* $\{v_i\}$ in V has a limit v that is an element of V. The Cauchy sequence $\{v_i\}$ means that $\|v_i - v_j\| \to 0$ as $i, j \to \infty$, and completeness says that $\|v_i - v\| \to 0$ as $i \to \infty$. A normed linear space $(V, \|\cdot\|)$ is called a *Banach space* if it is complete with respect to the norm $\|\cdot\|$. For $1 \leq q \leq \infty$, the space $L^q(\Omega)$ is a Banach space (Adams, 1975).

There are several useful inequalities that hold for functions in $L^q(\Omega)$. We state them without proof (Adams, 1975).

Hölder's inequality: For $1 \leq q, q' \leq \infty$ such that $1/q + 1/q' = 1$, it holds that

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} \qquad \forall u \in L^q(\Omega),\ v \in L^{q'}(\Omega) . \tag{1.29}$$

When $q = q' = 2$, this inequality is also called *Cauchy's* or *Schwarz's inequality*:

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \qquad \forall u, v \in L^2(\Omega) . \tag{1.30}$$

The triangle inequality applied to $L^q(\Omega)$ is referred to as *Minkowski's inequality*:

$$\|u + v\|_{L^q(\Omega)} \leq \|u\|_{L^q(\Omega)} + \|v\|_{L^q(\Omega)} \qquad \forall u, v \in L^q(\Omega) . \tag{1.31}$$

1.2.2 Weak Derivatives

We introduce the notation

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}} ,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index (called a d-tuple), with $\alpha_1, \alpha_2, \ldots, \alpha_d$ nonnegative integers, and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$ is the length of α. This notation indicates a partial derivative of v. For example, as $d = 2$, a second partial derivative can be written as $D^\alpha v$ with $\alpha = (2, 0)$, $\alpha = (1, 1)$, or $\alpha = (0, 2)$.

In calculus, derivatives of a function are defined pointwise. The variational formulation in the finite element method is given globally, i.e., in terms of integrals on Ω. Pointwise values of derivatives are not needed; only derivatives that can be interpreted as functions in Lebesgue spaces are used. Hence it is natural to introduce a global definition of *derivative* more suitable to the Lebesgue spaces.

For a continuous function v defined on Ω, the *support* of v is the closure of the (open) set $\{v : v(\mathbf{x}) \neq 0, \mathbf{x} \in \Omega\}$. If this set is *compact* (i.e., bounded), then v is called to have *compact support* in Ω. When Ω is bounded, it is equivalent to saying that v vanishes in a neighborhood of the boundary Γ of Ω.

For $\Omega \subset \mathbb{R}^d$, indicate by $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$ (the linear space of functions infinitely differentiable) functions that have compact support in Ω. We use the space $\mathcal{D}(\Omega)$ to introduce the concept of weak (generalized) derivatives. For this, we need the following function space:

$$L^1_{loc}(\Omega) = \{v : v \in L^1(K) \text{ for any compact } K \text{ inside } \Omega\} .$$

Note that $L^1_{loc}(\Omega)$ contains all of $C^0(\Omega)$ (continuous functions in Ω). Functions in $L^1_{loc}(\Omega)$ can behave arbitrarily badly near the boundary. With $\text{dist}(\mathbf{x}, \Gamma)$ denoting the distance from $\mathbf{x}$ to Γ, the function $e^{e^{1/\text{dist}(\mathbf{x},\Gamma)}} \in L^1_{loc}(\Omega)$, for example.

A function $v \in L^1_{loc}(\Omega)$ is said to have a *weak derivative*, $D^\alpha_w v$, if there is a function $u \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u(\mathbf{x})\varphi(\mathbf{x})\ d\mathbf{x} = (-1)^{|\alpha|}\int_\Omega v(\mathbf{x})D^\alpha\varphi(\mathbf{x})\ d\mathbf{x} \qquad \forall\varphi\in\mathcal{D}(\Omega)\ .$$

If such a function u does exist, we write $D_w^\alpha v = u$.

For any multi-index α, if $v \in C^{|\alpha|}(\Omega)$, then the weak derivative $D_w^\alpha v$ exists and equals $D^\alpha v$ (cf. Exercise 1.13). Consequently, we will ignore the difference in the definition of D_w^α and D^α. Namely, if classical derivatives do not exist, the differentiation symbol D^α will refer to weak derivatives.

Example. We consider a simple example with $d = 1$ and $\Omega = (-1, 1)$. Let $v(x) = 1 - |x|$. Then $D^1 v$ exists and is given by

$$u(x) = \begin{cases} -1 & \text{if } x > 0\ , \\ 1 & \text{if } x < 0\ . \end{cases}$$

In fact, for $\varphi\in\mathcal{D}(\Omega)$, an application of integration by parts yields

$$\begin{aligned}
&\int_{-1}^1 v(x)\frac{d\varphi}{dx}(x)\ dx \\
&= \int_{-1}^0 v(x)\frac{d\varphi}{dx}(x)\ dx + \int_0^1 v(x)\frac{d\varphi}{dx}(x)\ dx \\
&= [v\varphi]_{-1}^0 - \int_{-1}^0 (+1)\varphi(x)\ dx + [v\varphi]_0^1 - \int_0^1 (-1)\varphi(x)\ dx \\
&= -\int_{-1}^1 u(x)\varphi(x)\ dx\ ,
\end{aligned}$$

since v is continuous at 0.

Note that v is not differentiable at 0 in the classical sense. However, its first weak derivative exists. One can show that its higher order derivative $D^i v$ does not exist for $i > 2$ (cf. Exercise 1.14).

1.2.3 Sobolev Spaces

We now use weak derivatives to generalize the Lebesgue spaces introduced in Sect. 1.2.1.

For $r = 1, 2, \ldots$ and $v \in L^1_{loc}(\Omega)$, assume that the weak derivatives $D^\alpha v$ exist for all $|\alpha| \le r$. We define the *Sobolev norm*

$$\|v\|_{W^{r,q}(\Omega)} = \left(\sum_{|\alpha|\le r}\|D^\alpha v\|^q_{L^q(\Omega)}\right)^{1/q}\ ,$$

if $1 \le q < \infty$. For $q = \infty$, define

$$\|v\|_{W^{r,\infty}(\Omega)} = \max_{|\alpha|\le r}\|D^\alpha v\|_{L^\infty(\Omega)}\ .$$

The *Sobolev spaces* are defined by

$$W^{r,q}(\Omega) = \left\{v\in L^1_{loc}(\Omega) : \|v\|_{W^{r,q}(\Omega)} < \infty\right\}, \qquad 1\le q\le\infty\ .$$

One can check that $\|\cdot\|_{W^{r,q}(\Omega)}$ is a norm; moreover, the Sobolev space $W^{r,q}(\Omega)$ is a Banach space (Adams, 1975).

We denote by $W_0^{r,q}(\Omega)$ the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{r,q}(\Omega)}$.

For $\Omega\subset\mathbb{R}^d$ with smooth boundary and $v\in W^{1,q}(\Omega)$, the restriction to the boundary Γ, $v|_\Gamma$, can be interpreted as a function in $L^q(\Gamma)$ (Adams, 1975), $1\le q\le\infty$. This does not assert that pointwise values of v on Γ make sense. For $q = 2$, for example, $v|_\Gamma$ is only square integrable on Γ. Using this property, the space $W_0^{r,q}(\Omega)$ can be characterized as

$$W_0^{r,q}(\Omega) = \left\{v\in W^{r,q}(\Omega) : D^\alpha v|_\Gamma = 0 \text{ in } L^2(\Gamma),\ |\alpha| < r\right\}\ .$$

For later applications, the *seminorms* will be used:

$$\begin{aligned}
|v|_{W^{r,q}(\Omega)} &= \left(\sum_{|\alpha|=r}\|D^\alpha v\|^q_{L^q(\Omega)}\right)^{1/q}\ , \qquad 1\le q<\infty\ , \\
|v|_{W^{r,\infty}(\Omega)} &= \max_{|\alpha|=r}\|D^\alpha v\|_{L^\infty(\Omega)}\ .
\end{aligned}$$

Furthermore, for $q = 2$, we will utilize the symbols

$$H^r(\Omega) = W^{r,2}(\Omega),\ H_0^r(\Omega) = W_0^{r,2}(\Omega), \qquad r = 1, 2, \ldots\ .$$

That is, the functions in $H^r(\Omega)$, together with their derivatives $D^\alpha v$ of order $|\alpha|\le r$, are square integrable in Ω. Note that $H^0(\Omega) = L^2(\Omega)$.

The Sobolev spaces $W^{r,q}(\Omega)$ have a number of important properties. Given the indices defining these spaces, it is natural that there are inclusion relations to provide some type of ordering among them. We list a couple of inclusion relations; see Exercise 1.15.

For nonnegative integers r and k such that $r\le k$, it holds that

$$W^{k,q}(\Omega)\subset W^{r,q}(\Omega), \qquad 1\le q\le\infty\ . \tag{1.32}$$

In addition, when Ω is bounded,

$$W^{r,q'}(\Omega)\subset W^{r,q}(\Omega), \qquad 1\le q\le q'\le\infty\ , \tag{1.33}$$

for $r = 1, 2, \ldots$.

1.2.4 Poincaré's Inequality

We show an important inequality which will be heavily used in this book, *Poincaré's inequality*. It is sometimes called Poincaré-Friedrichs' inequality or simply Friedrichs' inequality.

Before introducing this inequality in its general form, we first consider one dimension. For any $v \in C_0^\infty(I)$ ($I = (0, 1)$, the unit interval), because $v(0) = 0$, we see that

$$v(x) = \int_0^x \frac{dv(y)}{dy}\, dy .$$

Consequently, by Cauchy's inequality (1.10), we have

$$|v(x)| \le \int_0^1 \left|\frac{dv(y)}{dy}\right| dy \le \left(\int_0^1 dy\right)^{1/2} \left(\int_0^1 \left(\frac{dv}{dy}\right)^2 dy\right)^{1/2} = \left(\int_0^1 \left(\frac{dv}{dy}\right)^2 dy\right)^{1/2},$$

which, by squaring and integrating over I, yields

$$\|v\|_{L^2(I)} \le |v|_{H^1(I)} .$$

Because $C_0^\infty(I)$ is dense in $H_0^1(I)$, we see that

$$\|v\|_{L^2(I)} \le |v|_{H^1(I)} \qquad \forall v \in V = H_0^1(I) . \tag{1.34}$$

This is Poincaré's inequality in one dimension.

We can extend this argument to the case where Ω is a d-dimensional cube: $\Omega = \{(x_1, x_2, \dots, x_d) : 0 < x_i < l,\ i = 1, 2, \dots, d\}$, where $l > 0$ is a real number. Again, since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove Poincaré's inequality for $v \in C_0^\infty(\Omega)$. Then we see that

$$v(x_1, x_2, \dots, x_d) = v(0, x_2, \dots, x_d) + \int_0^{x_1} \frac{\partial v}{\partial x_1}(y, x_2, \dots, x_d)\, dy .$$

Because the boundary term vanishes, it follows from Cauchy's inequality (1.10) that

$$|v(\mathbf{x})|^2 \le \int_0^{x_1} dy \int_0^{x_1} \left|\frac{\partial v}{\partial x_1}(y, x_2, \dots, x_d)\right|^2 dy \le l \int_0^l \left|\frac{\partial v}{\partial x_1}(y, x_2, \dots, x_d)\right|^2 dy .$$

Integrating over Ω implies

$$\|v\|_{L^2(\Omega)} \le l\, |v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega) . \tag{1.35}$$

For a general open set $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary, if $v \in H^1(\Omega)$ vanishes on a part of the boundary Γ with this part having positive $(d-1)$-dimensional measure, then there is a positive constant C, depending only on Ω, such that (Adams, 1975)

$$\|v\|_{L^2(\Omega)} \le C\, |v|_{H^1(\Omega)} . \tag{1.36}$$

If Ω is bounded, this inequality implies that the seminorm $|\cdot|_{H^1(\Omega)}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $H_0^1(\Omega)$. In general, an induction argument can be used to show that $|\cdot|_{H^r(\Omega)}$ is equivalent to $\|\cdot\|_{H^r(\Omega)}$ in $H_0^r(\Omega)$, $r = 1, 2, \dots$.

1.2.5 Duality and Negative Norms

Let V be a Banach space. A mapping $L : V \to \mathbb{R}$ is called a *linear functional* if

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v), \quad \alpha, \beta \in \mathbb{R},\ u,\ v \in V .$$

We say that L is *bounded* in the norm $\|\cdot\|_V$ if there is a constant $\tilde{L} > 0$ such that

$$|L(v)| \le \tilde{L} \|v\|_V \qquad \forall v \in V .$$

The set of bounded linear functionals on V is termed the *dual space* of V, and is denoted by V'.

A bounded linear functional L is actually *Lipschitz continuous* (and thus continuous); i.e.,

$$|L(v) - L(w)| = |L(v - w)| \le \tilde{L} \|v - w\|_V \qquad \forall v, w \in V .$$

Conversely, a continuous linear functional is also bounded. In fact, if it is not bounded, there is a sequence $\{v_i\}$ in V such that $|L(v_i)|/\|v_i\|_V \ge i$. Setting $w_i = v_i/(i\|v_i\|_V)$, we see that $|L(w_i)| \ge 1$ and $\|w_i\|_V = 1/i$. Then $w_i \to 0$ as $i \to \infty$, which, together with continuity of L, implies $L(w_i) \to 0$ as $i \to \infty$. This contradicts with $|L(w_i)| \ge 1$.

For $L \in V'$, define

$$\|L\|_{V'} = \sup_{0 \ne v \in V} \frac{L(v)}{\|v\|_V} .$$

Since L is bounded, this quantity is always finite. In fact, it induces a norm on V', called the *dual norm* (cf. Exercise 1.16), and V' is a Banach space with respect to it (Adams, 1975).

Let us consider the dual space of $L^q(\Omega)$, $1 \le q < \infty$. For $f \in L^{q'}(\Omega)$, where $1/q + 1/q' = 1$, set

$$L(v) = \int_\Omega f(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}, \qquad v \in L^q(\Omega) .$$

It follows from Hölder's inequality (1.29) that L is bounded in the L^q-norm:

$$|L(v)| \le \|f\|_{L^{q'}(\Omega)} \|v\|_{L^q(\Omega)}, \qquad v \in L^q(\Omega) .$$

Thus every function $f \in L^{q'}(\Omega)$ can be viewed as a bounded linear functional on $L^q(\Omega)$. Due to the Riesz Representation Theorem (cf. Sect. 1.3.1), all bounded linear functionals on $L^q(\Omega)$ arise in this form, so $L^{q'}(\Omega)$ can be viewed as the dual space of $L^q(\Omega)$. The number q' is often termed the *dual index* of q.

For $1 \le q \le \infty$ and a positive integer r, the dual space of the Sobolev space $W^{r,q}(\Omega)$ is indicated by $W^{-r,q'}(\Omega)$, where q' is the dual index of q. The norm on $W^{-r,q'}(\Omega)$ is defined via *duality*:

$$\|L\|_{W^{-r,q'}(\Omega)} = \sup_{0 \neq v \in W^{r,q}(\Omega)} \frac{L(v)}{\|v\|_{W^{r,q}(\Omega)}}, \qquad L \in W^{-r,q'}(\Omega) .$$

1.3 Abstract Variational Formulation

The introductory finite element method discussed in Sect. 1.1 will be written in an abstract formulation in this section. We first provide this formulation and its theoretical analysis, and then give several concrete examples. These examples will utilize the Sobolev spaces introduced in Sect. 1.2, particularly, the spaces $H^r(\Omega)$, $r = 0, 1, 2, \ldots$.

1.3.1 An Abstract Formulation

A linear space V, together with an inner product $(\cdot,\cdot)$ defined on it, is called an *inner product* space and is represented by $\big(V, (\cdot,\cdot)\big)$. With the inner product $(\cdot,\cdot)$, there is an associated norm defined on V:

$$\|v\| = \sqrt{(v,v)}, \qquad v \in V .$$

Hence an inner product space can be always made to be a normed linear space. If the corresponding normed linear space $\big(V, \|\cdot\|\big)$ is complete, then $\big(V, (\cdot,\cdot)\big)$ is termed a *Hilbert space*.

The space $H^r(\Omega)$ $(r = 1, 2, \ldots)$, with the inner product

$$(u,v)_{H^r(\Omega)} = \sum_{|\alpha| \le r} \int_\Omega D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x})\, d\mathbf{x}, \quad u,\ v \in H^r(\Omega)$$

and the corresponding norm $\|\cdot\|_{H^r(\Omega)}$, is a Hilbert space (Adams, 1975).

Suppose that V is a Hilbert space with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|_V$. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a *bilinear form* in the sense that

$$a(u, \alpha v + \beta w) = \alpha a(u,v) + \beta a(u,w) ,$$
$$a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w) ,$$

for $\alpha, \beta \in \mathbb{R}$, $u, v, w \in V$. Also, assume that $L : V \to \mathbb{R}$ is a linear functional. We define the functional $F : V \to \mathbb{R}$ by

$$F(v) = \frac{1}{2} a(v,v) - L(v), \qquad v \in V .$$

We now consider the abstract *minimization problem*

$$\text{Find } p \in V \text{ such that } F(p) \le F(v) \qquad \forall v \in V , \tag{1.37}$$

and the abstract *variational problem*

$$\text{Find } p \in V \text{ such that } a(p,v) = L(v) \qquad \forall v \in V . \tag{1.38}$$

To analyze (1.37) and (1.38), we need some properties of a and L:

- $a(\cdot,\cdot)$ is *symmetric* if

$$a(u,v) = a(v,u) \qquad \forall u,\ v \in V . \tag{1.39}$$

- $a(\cdot,\cdot)$ is *continuous* or *bounded* in the norm $\|\cdot\|_V$ if there is a constant $a^* > 0$ such that

$$|a(u,v)| \le a^* \|u\|_V \|v\|_V \qquad \forall u,\ v \in V . \tag{1.40}$$

- $a(\cdot,\cdot)$ is *V-elliptic* or *coercive* if there exists a constant $a_* > 0$ such that

$$|a(v,v)| \ge a_* \|v\|_V^2 \qquad \forall v \in V . \tag{1.41}$$

- L is bounded in the norm $\|\cdot\|_V$:

$$|L(v)| \le \tilde{L} \|v\|_V \qquad \forall v \in V . \tag{1.42}$$

The following theorem is needed in the proof of Theorem 1.1 below (Conway, 1985).

Theorem (Riesz Representation Theorem). *Let H be a Hilbert space with the scalar product $(\cdot,\cdot)_H$. Then, for any continuous linear functional $\mathcal{L}$ on H there is a unique $u \in H$ such that*

$$\mathcal{L}(v) = (u,v)_H .$$

We now prove the next theorem.

Theorem 1.1 (Lax-Milgram). *Under assumptions* (1.39)–(1.42), *problem* (1.38) *has a unique solution $p \in V$, which satisfies the bound*

$$\|p\|_V \le \frac{\tilde{L}}{a_*} . \tag{1.43}$$

Proof. Since the bilinear form a is symmetric and V-elliptic, it induces a scalar product in V:

$$[u,v] = a(u,v), \qquad u,\ v \in V .$$