THE CO-AREA FORMULA

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ABSTRACT. This surveys the co-area formula from its most elementary incarnation to the most sophisticated version. The main goal is to isolate the main ideas and describe how they fit in the general architecture of this result.

CONTENTS

INTRODUCTION

At its core, the co-area formula is a sophisticated version of Fubini's Theorem relating a double integral to an iterated integral. From a more modern point of view, the coarea formula describes the integration along the fibers of a Lipschitz map between rectifiable sets.

From Fubini to integration along fibers is a long road and the goal of these notes is to describe the main characters we meet along this road.

In Section 1 we consider the simplest case and we answer the simplest question: how do we integrate along the fibers of a surjective linear map between Euclidean spaces. We show that, up to orthogonal changes of coordinates this reduces to Fubini's Theorem but the changes in coordinates introduce an important character in the story namely, the Jacobian of a linear map. We spend most of this section analyzing this concept.

Submersions between differentiable manifolds are locally equivalent to surjective linear maps. In Section 2 we explain how to integrate along the fibers of a C^1-submersion between Riemann manifolds. This reduces easily to the linear case via partitions of unity.

If $F : X \to Y$ is an arbitrary C^1-map between differentiable manifolds then some of the fibers need not be smooth manifolds and we have to be careful in describing what measures we use to integrate along such fibers. It turns out that the correct choice is that of Hausdorff measures and in Section 3 we survey a few facts about these measures.

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Notes for the "*Blue collar seminar on geometric integration theory*" Many thanks to *Mattia Luchese* for fixing the many typos and radically rewriting Section 6.

In Section 4 we describe the co-area formula for arbitrary C^1-maps between Riemann manifolds. Our presentation is greatly inspired from [1, Sec. 13.4]. However, the story does not end here.

The Lipschitz maps between Riemann manifolds are differentiable almost everywhere and in Section 5 we explain how to integrate along the fibers of a Lipschitz map $\Phi : X \to Y$ between Riemann manifolds X, Y. Finally we relax our assumptions even more. Is there a co-area formula when X and Y are not differentiable everywhere, but only almost everywhere?

It turns out that such an extension is possible and the correct framework is that of rectifiable sets. This is discussed in the last section.

These started as notes for a talk at a working seminar. There are many presentations of the co-area formula and I am partial to the approach in [1]. The main goal of the notes is educational, to understand why and how the co-area formula works. In particular I took great pains to describe the metamorphosis of this result, from its simplest incarnation to the most sophisticated one, in the process trying to highlight the new concepts and ideas that allow the jumps to higher and higher levels of generality.

Acknowledgments. The initial version had many typos and the presentation of the co-area formula for rectifiable sets was rather negligent, to put it charitably. I am grateful to *Mattia Luchese* for volunteering to fix these problems. I especially want to mention Theorem 6.8, the most general version of the co-area formula. This is due to him and, in this form, it seems to be new.

1. Fubini theorem and the Jacobian of a linear map

Recall Fubini's theorem. Suppose φ is a integrable function on $\mathbb{R}^{n+k}$. Then

$$\begin{aligned}&\int_{\mathbb{R}^{n+k}} \varphi(x^1,\dots,x^{n+k})dx^1\cdots dx^{n+k}\\ &=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^k}\varphi(x^1,\dots,x^n,x^{n+1},\dots,x^{n+k})dx^{n+1}\cdots dx^{n+k}\right)dx^1\cdots dx^n.\end{aligned}$$

We can reformulate this as follows. Set

$$\boldsymbol{y}=(x^1,\dots,x^n),\;\;\boldsymbol{x}=(x^{n+1},\dots,x^{n+k})$$

and define $A:\mathbb{R}^{n+k}\to\mathbb{R}^n$, $(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{y}$. Then

$$\int_{\mathbb{R}^{n+k}}\varphi(\boldsymbol{x},\boldsymbol{y})|dV_{n+k}(\boldsymbol{x},\boldsymbol{y})|=\int_{\mathbb{R}^n}\left(\int_{A^{-1}(\boldsymbol{y})}\varphi(\boldsymbol{x},\boldsymbol{y})|dV_k(\boldsymbol{x})|\right)|dV_n(\boldsymbol{y})|. \tag{1.1}$$

where $|dV_i|$ denotes the i-dimensional Lebesgue measure.

Consider now a slightly more general case of a linear map

$$A:\mathbb{R}^{n+k}\to\mathbb{R}^n,\;\;(x^1,\dots,x^n,x^{n+1},\dots,x^{n+k})\mapsto(y^1,\dots,y^n)=(\mu_1x^1,\dots,\mu_nx^n), \tag{1.2}$$

where $\mu_1,\dots,\mu_n$ are positive numbers. Applying the Fubini theorem we deduce

$$\begin{aligned}&\int_{\mathbb{R}^{n+k}}\mu_1\cdots\mu_n\varphi(x^1,\dots,x^{n+k})|dV_{n+k}(x^1,\dots,x^{n+k})|\\ &=\int_{\mathbb{R}^{n+k}}\varphi\Big(\frac{y^1}{\mu^1},\dots\frac{y^n}{\mu^n},x^{n+1},\dots,x^{n+k}\Big)dy^1\cdots dy^ndx^{n+1}\cdots dx^{n+k}\\ &=\int_{\mathbb{R}^n}\left(\int_{A^{-1}(\boldsymbol{y})}\varphi(\boldsymbol{x},\boldsymbol{y})|dV_k(\boldsymbol{x})|\right)|dV_n(\boldsymbol{y})|.\end{aligned} \tag{1.3}$$

But for the factor $\mu_1\cdots\mu_n$, the formulæ (1.1) and (1.3) look similar. To give an invariant meaning to this quantity we need to use the following elementary fact of linear algebra.

Lemma 1.1. *Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are* Euclidean *spaces, respectively of dimensions $n+k$ and n ($n,k\geq 0$), and $A:\boldsymbol{U}\to\boldsymbol{V}$ is a linear map. Then there exist Euclidean coordinates $x^1,\dots,x^{n+k}$ on $\boldsymbol{U}$, Euclidean coordinates $y^1,\dots,y^n$ on $\boldsymbol{V}$ and nonnegative numbers $\mu_1,\dots,\mu_n$ such that, in these coordinates the operator A is described by*

$$y^j=\mu_j x^j,\ \ 1\leq j\leq n.$$

The numbers $\mu_1^2,\dots,\mu_n^2$ are the eigenvalues of the positive symmetric operator $AA^*:\boldsymbol{V}\to\boldsymbol{V}$ so that

$$\mu_1\cdots\mu_n=J_A:=\sqrt{\det AA^*}.$$

In particular

$$A\ \ \textit{surjective}\Longleftrightarrow J_A\neq 0.$$

The quantity J_A is called the Jacobian *of the linear map A.*

Proof. Let $\boldsymbol{W}$ denote the orthogonal complement of $\ker A$ in $\boldsymbol{U}$. Denote by A_0 the restriction of A to W so that $A_0:\boldsymbol{W}\to\boldsymbol{V}$ is a linear isomorphism. Note that $\boldsymbol{W}$ coincides with the range of the adjoint operator $A^*:\boldsymbol{V}\to\boldsymbol{U}$ so that

$$A_0A_0^*=AA^*.$$

We want to find a linear isometry $R:\boldsymbol{V}\to\boldsymbol{W}$ such that the operator

$$B=A_0R:\boldsymbol{V}\to\boldsymbol{V}$$

is symmetric. Note that since R is an isometry we have $R^{-1}=R^*$. Moreover we have a commutative diagram

$$\begin{tikzcd}
\boldsymbol{W} \arrow[r, "A_0"] & \boldsymbol{V} \\
\boldsymbol{V} \arrow[u, "R"] \arrow[r, "B"'] & \boldsymbol{V} \arrow[u, "\mathbb{1}_{\boldsymbol{V}}"']
\end{tikzcd}$$

Note that $A_0A^*:\boldsymbol{V}\to\boldsymbol{V}$ is nonnegative and symmetric. We define

$$R:=A_0^*(A_0A_0^*)^{-1/2}:\boldsymbol{V}\to\boldsymbol{W}.$$

Let us show that R is indeed an isometry. Indeed, for any $\boldsymbol{v}\in\boldsymbol{V}$ we have

$$\begin{aligned}(R\boldsymbol{v},R\boldsymbol{v})&=\big(\,A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v},\,A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\big)=\big(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\,A_0A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\big)\\&=\big(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\,(A_0A_0^*)^{1/2}\boldsymbol{v}\,\big)=(\boldsymbol{v},\boldsymbol{v}).\end{aligned}$$

Clearly $A_0R=A_0A_0^*(A_0A_0^*)^{-1/2}=(A_0A_0^*)^{1/2}$ is symmetric. Now choose an orthonormal basis that diagonalizes B. Transport it via R to an orthonormal basis of $\boldsymbol{W}$. With respect to these bases of $\boldsymbol{W}$ and $\boldsymbol{V}$ the operator A is described by a diagonal matrix with entries consisting of the eigenvalues of $A_0R=(A_0A_0^*)^{1/2}$.

$\square$

Thus, we can rewrite (1.3) as

$$\int_{\mathbb{R}^{n+k}}J_A(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{x},\boldsymbol{y})|dV_{n+k}(\boldsymbol{x},\boldsymbol{y})|=\int_{\mathbb{R}^n}\left(\int_{A^{-1}(\boldsymbol{y})}\varphi(\boldsymbol{x},\boldsymbol{y})|dV_k(\boldsymbol{x})|\right)|dV_n(\boldsymbol{y})|.\tag{1.4}$$

Lemma 1.1 shows that (1.4) holds for any *surjective linear* map $\mathbb{R}^{n+k}\to\mathbb{R}^n$.

Proposition 1.2. *Suppose that* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are* Euclidean *spaces, respectively of dimensions* $n+k$ *and* n $(n,k\geq 0)$*, and* $A:\boldsymbol{U}\to\boldsymbol{V}$ *is a linear map. Then*

$$J_A=\frac{\mathrm{Vol}_n\big(\,A(\,B_1^{\boldsymbol{U}}\,)\,\big)}{\mathrm{Vol}_n\big(\,B_1^{\boldsymbol{V}}\,\big)}, \tag{1.5}$$

where Vol_n *denotes the* n*-dimensional Euclidean volume on* $\boldsymbol{V}$*,* $B_1^{\boldsymbol{U}}$ *denotes the unit ball in* $\boldsymbol{U}$ *and* $B_1^{\boldsymbol{V}}$ *the unit ball in* $\boldsymbol{V}$

Proof. Choose coordinates (x^i) on $\boldsymbol{U}$ and (y^j) on $\boldsymbol{V}$ as in Lemma 1.1. If A is not onto the result is obvious since, then $\dim A(\boldsymbol{U})<n$. If A is onto, then $A(\,B_1^{\boldsymbol{U}}\,)$ is isometric to the ellipsoid

$$E=\left\{\,x\in\mathbb{R}^n;\ \ \sum_{j=1}^n\frac{(x^j)^2}{\mu_j^2}\leq 1\,\right\}$$

where the numbers μ_j are as in Lemma 1.1. Observe that $\mathrm{Vol}_n(E)=\mu_1\cdots\mu_n$. □

It is convenient to give a more explicit algebraic description of J_A. This relies on the concept of Gram determinant. More precisely, given a collection of vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ in an Euclidean space $\boldsymbol{U}$ we define their *Gram determinant* (or *Gramian*) to be the quantity

$$\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n):=\det\Big(\,(\boldsymbol{u}_i,\boldsymbol{u}_j)_{\boldsymbol{U}}\,\Big)_{1\leq i,j\leq n},$$

where $(-,-)_{\boldsymbol{U}}$ denotes the inner product in $\boldsymbol{U}$. Geometrically, $\sqrt{\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)}$ is the n-dimensional volume of the parallelepiped spanned by the vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$,

$$P(\boldsymbol{w}_1,\dots,\boldsymbol{w}_n)=\Big\{\,\sum_{j=1}^n t_j\boldsymbol{w}_j;\ \ t_j\in[0,1]\,\Big\}.$$

Note that $\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=0$ iff the vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ are linearly dependent and $\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=1$ if the vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ form an orthonormal system.

Equivalently

$$\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=\big(\,\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_n,\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_n\,\big)_{\Lambda^n\boldsymbol{U}}$$

where $(-,-)_{\Lambda^n U}$ denotes the inner product on $\Lambda^n\boldsymbol{U}$ induced by the inner product in $\boldsymbol{U}$.

Lemma 1.3. *Let* $A:\boldsymbol{U}\to\boldsymbol{V}$ *be as in Lemma 1.1. Fix a basis* $\boldsymbol{f}_{n+1},\dots,\boldsymbol{f}_{n+k}$ *of* $\boldsymbol{U}_0:=\ker A$ *and vectors* $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ *such that* $A\boldsymbol{u}_1,\dots,A\boldsymbol{u}_n$ *span* $\boldsymbol{V}$*. Then*

$$J_A^2=\frac{\mathbb{G}(A\boldsymbol{u}_1,\dots A\boldsymbol{u}_n)\mathbb{G}(\boldsymbol{f}_{n+1},\dots,\boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n,\boldsymbol{f}_{n+1},\dots,\boldsymbol{f}_{n+k})}. \tag{1.6}$$

Proof. We first prove the result when $\dim\boldsymbol{U}=\dim\boldsymbol{V}$. In this case the collection $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ is a basis of $\boldsymbol{U}$. Fix an orthonormal basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ of $\boldsymbol{U}$ denote by $T:\boldsymbol{U}\to\boldsymbol{U}$ the linear operator

$$\boldsymbol{e}_j\mapsto\boldsymbol{u}_j.$$

Then

$$\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=\det T^*T,$$

$$\mathbb{G}(A\boldsymbol{u}_1,\dots A\boldsymbol{u}_n)=\det((AT)^*(AT))=|\det T^*|\det AA^*|\det T|=J_A^2\det TT^*.$$

To deal with the general case, we denote by P_0 the orthogonal projection onto $\boldsymbol{U}_0$. Now define

$$\widehat{A}:\boldsymbol{U}\to\widehat{\boldsymbol{V}}:=\boldsymbol{V}\oplus\boldsymbol{U}_0,\ \ \boldsymbol{u}\mapsto A\boldsymbol{u}\oplus P_0\boldsymbol{u}.$$

we equip $\widehat{\boldsymbol{V}}$ with the product Euclidean structure.

Let us observe that

$$J_A = J_{\widehat{A}}.$$

Indeed, with respect to the (orthogonal) direct sum decomposition $\widehat{\boldsymbol{V}} = \boldsymbol{V} \oplus \boldsymbol{U}_0$ the operator $\widehat{A}\widehat{A}^*$ has the block decomposition

$$\widehat{A}\widehat{A}^* = \begin{bmatrix} AA^* & 0 \\ * & \mathbb{1}_{\boldsymbol{U}_0} \end{bmatrix}$$

so that

$$\det \widehat{A}\widehat{A}^* = \det AA^*.$$

Observe that in $\Lambda^{n+k}(\boldsymbol{V} \oplus \boldsymbol{U}_0)$ we have the equality

$$\hat{A}\boldsymbol{u}_1 \wedge \cdots \hat{A}\boldsymbol{u}_n \wedge \boldsymbol{f}_{n+1} \wedge \cdots \wedge \boldsymbol{f}_{n+k} = A\boldsymbol{u}_1 \wedge \cdots A\boldsymbol{u}_n \wedge \boldsymbol{f}_{n+1} \wedge \cdots \wedge \boldsymbol{f}_{n+k}$$

so that

$$\begin{aligned} \mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_n, \widehat{A}\boldsymbol{f}_{n+1}, \dots, \widehat{A}\boldsymbol{f}_{n+k}) &= \mathbb{G}(A\boldsymbol{u}_1, \dots, A\boldsymbol{u}_n, \boldsymbol{f}_{n+1}, \dots, \boldsymbol{f}_{n+k}) \\ &= \mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_n)\mathbb{G}(\boldsymbol{f}_{n+1}, \dots, \boldsymbol{f}_{n+k}). \end{aligned}$$

Now apply the first part of the proof to deduce that

$$J_A^2 = J_{\widehat{A}}^2 = \frac{\mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_n, \widehat{A}\boldsymbol{f}_{n+1}, \dots, \widehat{A}\boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n, \boldsymbol{f}_{n+1}, \dots, \boldsymbol{f}_{n+k})} = \frac{\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_n)\mathbb{G}(\boldsymbol{f}_{n+1}, \dots, \boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n, \boldsymbol{f}_{n+1}, \dots, \boldsymbol{f}_{n+k})}.$$

□

Suppose now that M and N are C^1 manifolds of dimensions respectively $n+k$ and n (k, $n \geq 0$), equipped with Riemann metrics g_M and g_N. We denote by $|dV_M|$ and $|dV_N|$ the volume densities induced by g_M and respectively g_N.

2. The co-area formula for submersions

A submersion between differentiable manifolds is locally equivalent with a surjective linear map. The considerations in the previous sections extend without much effort to this case. Suppose now that X and Y are C^1 manifolds of dimensions $n+k$ and respectively k, $n \geq 0$ equipped with Riemann metrics g_X and g_Y. We denote by $|dV_X|$ and $|dV_Y|$ the volume densities induced by g_X and respectively g_Y.

Theorem 2.1 (The co-area formula: version 1)**.** *Suppose that $F : X \to Y$ is a C^1-map such that for any $\boldsymbol{p} \in M$ the differential $D_{\boldsymbol{p}}F : T_{\boldsymbol{p}}X \to T_{F(\boldsymbol{p})}Y$ is surjective. We denote by $J_F(\boldsymbol{p})$ the Jacobian of this map. For any nonnegative function $\varphi : X \to \mathbb{R}$ which is measurable with respect to the measure defined by $|dV_X|$ we have*

$$\int_X J_F(\boldsymbol{p})\varphi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} \varphi(\boldsymbol{p})|dV_{F^{-1}(\boldsymbol{q})}(\boldsymbol{p})| \right) |dV_Y(\boldsymbol{q}), \tag{2.1}$$

where $|dV_{F^{-1}(\boldsymbol{q})}|$ denotes the volume density on the fiber $F^{-1}(\boldsymbol{q})$ induced by the restriction of g_X to $F^{-1}(\boldsymbol{q})$.

Proof. We consider first the case when X is an open subset of $\mathbb{R}^{n+k}$ with coordinates $(x^1, \dots, x^{n+k})$ equipped with a C^1-metric g_X, Y is an open subset of $\mathbb{R}^k$ with coordinates $(y^1, \dots, y^k)$ equipped with a metric g_Y and the map F is given by

$$y^j = x^j, \;\; j = 1, \dots, k.$$

We have

$$
\begin{aligned}
|dV_X| &= \sqrt{\mathbb{G}(\partial_{x^1},\dots,\partial_{x^{n+k}})}|dx^1\cdots dx^{n+k}|\\
&= \underbrace{\sqrt{\mathbb{G}(\partial_{x^1},\dots,\partial_{x^{n+k}})}}_{=:\rho_X}|dy^1\cdots dy^k dx^{k+1}\cdots dx^{n+k}|,\\
|dV_{F^{-1}(q)}| &= \underbrace{\sqrt{\mathbb{G}(\partial_{x^{k+1}},\dots,\partial_{x^{n+k}})}}_{=:\rho_F}|dx^{k+1}\cdots dx^{n+k}|,
\end{aligned}
$$

where the subscript X indicates that the inner product in the definition of the above Gramm determinants is the one determined by the Riemann metric on X. Similarly

$$
|dV_Y| = \underbrace{\sqrt{G_Y(\partial_{y^1},\dots,\partial_{y^k}))}}_{=:\rho_Y}|dy^1\cdots dy^k| = \sqrt{G_Y(DF\partial_{x^1},\dots,DF\partial_{x^k}))}|dy^1\cdots dy^k|.
$$

Using the Fubini theorem we deduce that for any nonnegative, measurable function $\phi : X\to\mathbb{R}$ we have

$$
\int_X \rho_Y\phi\rho_X|dy^1\cdots dy^k dx^{k+1}\cdots dx^{n+k}| = \int_Y\left(\int_{F^{-1}(\boldsymbol{y})}\rho_X\phi|dx^{k+1}\cdots dx^{n+k}|\right)\rho_Y|dy^1\cdots dy^k|
$$

$$
= \int_Y\left(\int_{F^{-1}(\boldsymbol{y})}\frac{\rho_X}{\rho_F}\phi\rho_F|dx^{k+1}\cdots dx^{n+k}|\right)|dV_Y(\boldsymbol{y})| = \int_Y\left(\int_{F^{-1}(\boldsymbol{y})}\frac{\rho_X}{\rho_F}\phi|dV_{F^{-1}(\boldsymbol{y})}|\right)|dV_Y(\boldsymbol{y})|.
$$

Suppose that above $\rho_Y\phi = J_F\varphi$, i.e., $\phi=\frac{J_F}{\rho_Y}\varphi$. Then the above equality can be rewritten

$$
\int_X J_F(\boldsymbol{x})\varphi(\boldsymbol{x})\,|dV_X(\boldsymbol{x})| = \int_Y\left(\int_{F^{-1}(\boldsymbol{y})}\frac{\rho_X J_F}{\rho_F\rho_Y}\varphi|dV_{F^{-1}(\boldsymbol{y})}|\right)|dV_Y(\boldsymbol{y})|.
$$

The co-area formula is proved once we show that

$$
\frac{\rho_X J_F}{\rho_F\rho_Y} = 1,\ \ \text{i.e.,}\ \ J_F = \frac{\rho_Y\rho_F}{\rho_X}.
$$

The last equality follows from (1.6).

The general case of the co-area formula can be reduced to the special case via partition of unity and the implicit function theorem. □

Corollary 2.2. *Let X, Y and $F : X\to Y$ be as in Theorem 2.1. Then for any measurable function $\phi : X\to\mathbb{R}$ we have*

$$
\int_X \phi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_Y\left(\int_{F^{-1}(\boldsymbol{q})}\frac{\phi(\boldsymbol{p})}{J_F(\boldsymbol{p})}|dV_{F^{-1}(\boldsymbol{q})}(\boldsymbol{p})|\right)|dV_Y(\boldsymbol{q}), \tag{2.2}
$$

as long as either side of the equality is finite.

Proof. Write $\phi=\phi^+-\phi^-$ and then apply (2.1) to $\varphi^\pm=\frac{\phi^\pm}{J_F}$. □

Corollary 2.3. *Suppose X is a C^1 manifold equipped with a C^1-metric g_X, and $f : X\to\mathbb{R}$ is a C^1 function with no critical points . Then for any measurable function $\phi : X\to\mathbb{R}$ we have*

$$
\int_X \phi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_{\mathbb{R}}\left(\int_{\{f=t\}}\frac{\phi(\boldsymbol{p})}{|\nabla f(\boldsymbol{p})|}|dV_{f^{-1}(t)}(\boldsymbol{p})|\right)dt. \tag{2.3}
$$

In particular, by setting $\phi = 1$ we deduce

$$\mathrm{vol}\,(X) = \int_{\mathbb{R}}\left(\int_{\{f=t\}}\frac{1}{|\nabla f(\boldsymbol{p})|}|dV_{f^{-1}(t)}(\boldsymbol{p})|\right)dt. \tag{2.4}$$

□

Example 2.4. We want to show how to use (2.4) to compute $\boldsymbol{\sigma}_n$, the "area" of the unit sphere

$$S^n = \Big\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^n;\ \ \sum_{j=0}^{n} x_j^2 = 1\Big\}.$$

Let S_*^n denote the unit sphere with the poles $x_0 = \pm 1$ removed. Then $\boldsymbol{\sigma}_n = \mathrm{vol}\,(S_*^n)$.

Consider $f : S_*^n \to \mathbb{R}$, $f(x_0, \dots, x_n) = x_0$. This function has no critical points on S_*^n. Let $\boldsymbol{p} \in S_*^n$ such that $f(\boldsymbol{p}) = x_0(\boldsymbol{p}) = t$. Denote by φ the angle between the radius $O\boldsymbol{p}$ and the x_0-axis. Note that

$$\cos\varphi = x_0 = t.$$

The gradient of f is the projection of ∂_{x_0} on the tangent plane $T_{\boldsymbol{p}}S^n$. We deduce that

$$|\nabla f(\boldsymbol{p})| = |\partial_{x_0}|\sin\varphi = (1-t^2)^{1/2}.$$

The level set $\{f = t\}$ is an $(n-1)$-dimensional sphere of radius $(1-t^2)^{1/2}$ and we deduce

$$\int_{\{f=t\}}\frac{1}{|\nabla f(\boldsymbol{p})|}|dV_{f^{-1}(t)}(\boldsymbol{p})| = (1-t^2)^{-1/2}\mathrm{vol}\,(f=t) = \boldsymbol{\sigma}_{n-1}(1-t^2)^{\frac{n-2}{2}}.$$

Hence

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_{n-1}\int_{-1}^{1}(1-t^2)^{\frac{n-2}{2}}dt = 2\boldsymbol{\sigma}_{n-1}\int_0^1(1-t^2)^{\frac{n-2}{2}}dt$$

$(t = \sqrt{s})$

$$= \boldsymbol{\sigma}_{n-1}\int_0^1(1-s)^{\frac{n}{2}-1}s^{\frac{1}{2}-1}ds =: B\Big(\frac{n}{2}, \frac{1}{2}\Big).$$

The integral

$$B(p,q) = \int_0^1(1-x)^{p-1}x^{q-1}dx,\ \ p, q > 0,$$

was computed by Euler who showed that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\frac{\boldsymbol{\sigma}_n}{\boldsymbol{\sigma}_{n-1}} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}.$$

Using the equalities $\boldsymbol{\sigma}_0 = 2$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we deduce

$$\boldsymbol{\sigma}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

We can obtain easily $\boldsymbol{\omega}_n$, the volume of the unit n-dimensional ball,

$$\boldsymbol{\omega}_n = \frac{1}{n}\boldsymbol{\sigma}_{n-1} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{2.5}$$

□

3. The Hausdorff measures

Suppose (X, d) is a separable metric space. Fix a nonnegative real number r. For any positive number δ and any set $E \subset X$ we set

$$\mathcal{H}^r_\delta(E) := \frac{\boldsymbol{\omega}_r}{2^r} \inf \left\{ \sum_{j \geq 1} (\operatorname{diam} B_j)^r; \;\; E \subset \bigcup_{j \geq 1} B_j, \;\; \operatorname{diam} B_j < \delta \right\}$$

(the B_j are arbitrary subsets of X). Note that if

$$0 < \delta_0 < \delta_1 \Rightarrow \mathcal{H}^r_{\delta_0}(E) \geq \mathcal{H}^r_{\delta_1}(E).$$

Thus the limit

$$\lim_{\delta \searrow 0} \mathcal{H}^r_\delta(E)$$

exists and we denote it by $\mathcal{H}^r$. The correspondence $E \mapsto \mathcal{H}^r(E)$ is an outer measure satisfying the Caratheodory condition, [6, Chap.12]

$$\operatorname{dist}(E_1, E_2) > 0 \Rightarrow \mathcal{H}^r(E_1 \cup E_2) = \mathcal{H}^r(E_1) + \mathcal{H}^r(E_2).$$

This implies, [6, Chap. 5], that any Borel set B is measurable with respect to $\mathcal{H}^r$, i.e.,

$$\mathcal{H}^r(E) = \mathcal{H}^r(E \cap B) + \mathcal{H}^r(E \setminus B), \;\; \forall E \subset X.$$

We denote by $\sigma_r(X)$ the set of $\mathcal{H}^r$-measurable subsets of X and continue to use the symbol $\mathcal{H}^r$, or $\mathcal{H}^r_X$, for the restriction of the outer measure $\mathcal{H}^r$ to $\sigma_r(X)$. The measure $\mathcal{H}^r$ is called the r-the Hausdorff measure.

Example 3.1. (a) If M is a C^1-manifold of dimension m equipped with a C^0- Riemann metric g that induces a metric space structure on M, then for any Borel set $B \subset M$ we have

$$\mathcal{H}^m_M(B) = \operatorname{vol}_g(B).$$

In particular, $\mathcal{H}^m_M$ coincides with the measure induced by the volume density determined by g. On $\mathbb{R}^n$, this also implies $\mathcal{L}^n_{\mathbb{R}^n} = \mathcal{H}^n_{\mathbb{R}^n}$, i.e. the Lebesgue measure and the Hausdorff measure coincide.
(b) If X, Y are locally compact metric spaces, $F : X \to Y$ is a Lipschitz map with Lipschitz constant $\leq L$, and $B \subset X$ is a Borel set with $\mathcal{H}^r(B) < \infty$, then $F(B)$ is $\mathcal{H}^r_Y$-measurable and

$$\mathcal{H}^r_Y\big(\, F(B)\,\big) \leq L^r \mathcal{H}^r_X(B) \qquad \text{for all } r > 0.$$

For proofs of the above statements (a) and (b) we refer to [6, Chap 12]. □

We have the following density result concerning Hausdorff measurable functions. For a proof we refer to [3, §4.3] or [5, §3.].

Theorem 3.2. *Suppose that X is a separable metric space $E \subset X$ is a $\mathcal{H}^m$-measurable set such that $\mathcal{H}^m(E) < \infty$. Then for for $\mathcal{H}^m$-almost any $x \in X \setminus E$ we have*

$$\underbrace{\limsup_{r \searrow 0} \frac{\mathcal{H}^m\big(\, B_r(x) \cap E\,\big)}{\boldsymbol{\omega}_m r^m}}_{=:\Theta^{*m}(E,x)} = 0.$$

Corollary 3.3. *Suppose M is an m-dimensional Riemann manifold and $S \subset M$ is a C^1 submanifold of dimension k. There exists a subset $S^* \subset S$ such that the following hold.*

- $\mathcal{H}^k(S \setminus S^*) = 0.$

- *For any $x \in S^*$ we have*
$$\lim_{r\searrow 0} \frac{\mathcal{H}^k(S\cap B_r(x))}{\mathcal{H}^k(M\cap B_r(x))} = 1.$$

Proof. We have
$$\mathcal{H}^m(S\cap B_r(x)) = \mathcal{H}^m(X\cap B_r(x)) - \mathcal{H}^m\big(\, S^c\cap B_r(x)\,\big).$$
From Theorem 3.2 we deduce that there exits a subset $S^*\subset S$ such that $\mathcal{H}^m(S\setminus S^*)=0$ and for any $x\in S^*$ we have
$$\Theta^{*m}(S^c,x)=0,\ \ \text{i.e.}\ \ \limsup_{r\searrow 0}\frac{\mathcal{H}^m(S^c\cap B_r(x))}{\boldsymbol{\omega}_m r^m}=0.$$
We deduce
$$\begin{aligned}\liminf_{r\searrow 0}\frac{\mathcal{H}^m(S\cap B_r(x))}{\boldsymbol{\omega}_m r^m} &= \liminf_{r\searrow 0}\frac{\mathcal{H}^m(X\cap B_r(x))}{\boldsymbol{\omega}_m r^m} - \limsup_{r\searrow 0}\frac{\mathcal{H}^m(S^c\cap B_r(x))}{\boldsymbol{\omega}_m r^m}\\ &= \liminf_{r\searrow 0}\frac{\mathcal{H}^m(X\cap B_r(x))}{\boldsymbol{\omega}_m r^m}.\end{aligned}$$
The desired conclusion follows by observing that
$$\lim_{r\searrow 0}\frac{\boldsymbol{\omega}_m r^m}{\mathcal{H}^m(X\cap B_r(x))} = 1 \geq \liminf_{r\searrow 0}\frac{\mathcal{H}^m(S\cap B_r(x))}{\mathcal{H}^m(X\cap B_r(x))}.$$
The result is now obvious. □

Theorem 3.4 (Eilenberg inequality). *Suppose (X,d_X) is a separable metric space and Y is a C^1 manifold of dimension k equipped with a C^0-Riemann metric g. Denote by $d_Y: Y\times Y\to\mathbb{R}$ the metric on Y induced by g. Let $F: X\to Y$ be a map satisfying the Lipschitz condition*
$$d_Y\big(F(x_1),F(x_2)\,\big)\leq Ld_X(x_1,x_2),\ \ \forall x_1,x_2\in X.$$
Then for any $m\geq k$ there exists a constant[1] *$C(m,k)>0$ such that for any Borel set $A\subset X$ we have*
$$\int_Y^* \mathcal{H}_X^{m-k}\big(A\cap F^{-1}(y)\,\big)d\mathcal{H}^k(y)\leq C(m,k)L^k\mathcal{H}^m(A),$$
where $\int^*$ denotes the upper Lebesgue integral. □

For a proof of this inequality we refer to [1, §13.3] or [3, §5.2.1]. The strategy behind the proof is identical to the strategy behind the proof of Lemma 4 described a bit later. As explained in [3, §5.2.1], this inequality implies the following technical result.

Corollary 3.5. *Let $F: X\to Y$ be as in Theorem 3.4. Then for any $m\geq k$ and any Borel subset $A\subset X$ the map*
$$Y\ni y\mapsto \mathcal{H}_X^{m-k}\big(A\cap F^{-1}(y)\,\big)\in[0,\infty]$$
is $\mathcal{H}_Y^k$-measurable. □

[1] We can choose $C(m,k)=\frac{\boldsymbol{\omega}_{m-k}\boldsymbol{\omega}_k}{\boldsymbol{\omega}_m}$

4. The co-area formula for arbitrary differential maps

We have now all the technical background needed to state and prove a more general co-area formula

Theorem 4.1 (The co-area formula: version 2)**.** *Suppose X and Y are connected, Riemann C^1-manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : X \to Y$ is a C^1-map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}^{n+k}_X$-measurable subset $A \subset X$ we have

$$\underbrace{\int_A J_F(x) d\mathcal{H}^{n+k}_X(x)}_{=:I(A)} = \underbrace{\int_Y \mathcal{H}^n_M\big(\, A \cap F^{-1}(y)\,\big) d\mathcal{H}^k_Y(y)}_{=:J(A)}. \tag{4.1}$$

Proof. When A contains no critical points of F, the result follows directly from Theorem 2.1 applied to $X \setminus \{J_F = 0\}$ instead of X and $\varphi = 1_A$, the indicator function of A. For a general A, we write $A = (A \cap \{J_F = 0\}) \cup (A \setminus \{J_F = 0\})$. The conclusion then follows from the following Sard-like result (which also guarantees measurability since $\mathcal{H}^n$ is complete). □

Lemma 4.2. *The following equality holds true:*

$$\int_Y \mathcal{H}^k_X\big(\,\{J_F = 0\} \cap F^{-1}(y)\,\big)\, d\mathcal{H}^n_Y(y) = 0.$$

In other words, $\mathcal{H}^k_X\big(\,\{J_F = 0\} \cap F^{-1}(y)\,\big) = 0$ for $\mathcal{H}^n_Y$-a.e. $y \in Y$.

Proof. Since X is σ-compact it suffices to prove that if C is a compact subset of X such that $J_F(x) = 0$, for any $x \in C$, then

$$\int_Y \mathcal{H}^n_X(C \cap F^{-1}(y)\,)\, d\mathcal{H}^k_Y(y) = 0.$$

We follow closely the proof of [1, Lemma 13.4.4].

Let us first observe that for any $\boldsymbol{p} \in C$ and any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(\boldsymbol{p})$ such that for any $0 < r < r_\varepsilon(x)$ we have

$$\mathcal{H}^k\big(\, F(\, B(\boldsymbol{p}, r)\,)\,\big) \leq \varepsilon L^{k-1} r^k. \tag{4.2}$$

Indeed, we have $\operatorname{rank} D_{\boldsymbol{p}}F \leq k-1$. The definition of the differential of F at x implies that, given a choice of coordinates x near $\boldsymbol{p}$ such that $x(\boldsymbol{p}) = 0$ we have

$$F(x) = F(0) + A_{\boldsymbol{p}}x + o(|x|), \;\; A_{\boldsymbol{p}} := D_{\boldsymbol{p}}F.$$

Hence, for any $\varepsilon > 0$, the set $F(\, B(\boldsymbol{p}, r)\,)$ is contained in a k-dimensional polydisk of the form $\mathbb{D}^{k-1}(F(\boldsymbol{p}), Lr) \times [-\varepsilon r, \varepsilon r]$ if r is sufficiently small, $r < r_\varepsilon(\boldsymbol{p})$. Above, $\mathbb{D}^{k-1}(y, R)$ indicates a $(k-1)$-disk of center y and radius R. Since C is compact we can assume that

$$r_\varepsilon := \inf_{\boldsymbol{p} \in C} r_\varepsilon(\boldsymbol{p}) > 0.$$

We deduce that

$$\mathcal{H}^k\big(\, F(S \cap C)\,\big) \leq \varepsilon L^{k-1} \operatorname{diam}(S)^k, \;\; \forall S \subset X, \;\; \operatorname{diam} S < \frac{1}{2} r_\varepsilon. \tag{4.3}$$

For any $s > 0$ we can find a countable cover of C in X by measurable sets $(X_i^s)_{i\geq 1}$ such that

$$\operatorname{diam}(X_i^s) < \frac{1}{s} \ \text{ and } \ \mathcal{H}^{n+k}(C) \geq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \sum_{i\geq 1} (\operatorname{diam} X_i^s)^{n+k} - \frac{1}{s}. \tag{4.4}$$

By definition

$$\mathcal{H}^n\big(C \cap f^{-1}(y)\big) \leq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s\to\infty} \sum_{i\geq 1} \big(\operatorname{diam} X_i^s \cap f^{-1}(y)\big)^n.$$

For any set $E \subset X$ we denote by φ_E the characteristic function of the closure of $F(E)$. We can then rewrite the above equality as

$$\mathcal{H}^n\big(C \cap f^{-1}(y)\big) \leq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s\to\infty} \sum_{i\geq 1} \big(\operatorname{diam} X_i^s\big)^n \varphi_{X_i^s}(y).$$

The Fatou lemma then implies

$$\int_Y^* \mathcal{H}^n\big(C \cap f^{-1}(y)\big)\, d\mathcal{H}_Y^k \leq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s\to\infty} \sum_{i\geq 1} \big(\operatorname{diam} X_i^s\big)^n \int_Y \varphi_{X_i^s}(y) d\mathcal{H}_Y^k.$$

Fix $\varepsilon > 0$. We deduce from (4.3) that for s sufficiently large, $s > s_\varepsilon$ we have

$$\int_Y \varphi_{X_i^s}(y) d\mathcal{H}_Y^k \leq \varepsilon L^{k-1} (\operatorname{diam} X_i^s)^k.$$

Hence

$$\begin{aligned}\int_Y^* \mathcal{H}^n\big(C \cap f^{-1}(y)\big)\, d\mathcal{H}_Y^k &\leq \varepsilon L^{k-1} \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s\to\infty} \sum_{i\geq 1} \big(\operatorname{diam} X_i^s\big)^{n+k} \\ &\overset{(4.4)}{\leq} \varepsilon L^{k-1} \left(\mathcal{H}^{n_k}(C) + \frac{1}{s_\varepsilon}\right).\end{aligned}$$

Now let $\varepsilon \to 0$. $\square$

Remark 4.3. The proof of the Eilenberg inequality follows an identical strategy with the inequality (4.2) replaced by the inequality

$$\mathcal{H}^k\big(F(S)\big) \leq C(m,k)(\dim S)^k$$

for any Borel set $S \subset X$ with sufficiently small diameter. $\square$

Corollary 4.4. *Let $F : X \to Y$ be as in Theorem 4.1. Then for any measurable function $\varphi : X \to \mathbb{R}$ we have*

$$\int_X \varphi(x) J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_Y \left(\int_{F^{-1}(y)} \varphi(x) d\mathcal{H}_X^n(x) \right) d\mathcal{H}_Y^k(y), \tag{4.5}$$

as soon as either side of the above equality is finite.

Proof. By Theorem 4.1 the equality (4.5) is true when φ is the characteristic function of a measurable subset of X. By linearity, (4.5) is true for linear combinations of such functions. We now observe that for any measurable nonnegative function φ we can find a sequence of simple functions $(\varphi_\nu)_{\nu\geq 1}$ that converges increasingly and almost everywhere to φ. Finally, the general case follows by decomposing a measurable function as a difference of two nonnegative ones $\varphi = \varphi^+ - \varphi^-$. $\square$

Corollary 4.5. *Suppose that $F : X \to Y$ is as in Theorem 4.1. We assume additionally that X and Y are* smooth, *oriented and F is also smooth. Denote by Y^* the set of regular values*[2] *of Y. When $y \in Y^*$ we orient the fiber $F^{-1}(y)$ using the fiber first convention*

$$\text{orientation}\,(X) = \text{orientation}\,F^{-1}(y) \wedge \text{orientation}\,(Y).$$

Then for any compactly supported C^1-form $\eta \in \Omega^n(X)$ the map

$$Y^* \ni y \mapsto \int_{F^{-1}(y)} \eta \in \mathbb{R}$$

is measurable and

$$\int_{Y^*} \left(\int_{F^{-1}(y)} \eta \right) dV_Y(y) = \int_X (\eta \wedge F^* dV_Y)(x), \tag{4.6}$$

where dV_X and dV_Y are the Riemannian volume forms on X and respectively Y. More generally, if $\alpha \in \Omega^{k+n}(X)$ is a compactly supported C^1-form then

$$\int_X \alpha = \int_{Y^*} \left(\int_{F^{-1}(y)} \frac{\alpha}{F^* dV_Y} \right) dV_Y(y), \tag{4.7}$$

where, along a regular fiber $F^{-1}(y)$, the Gelfand-Leray residue $\frac{\alpha}{F^*dV_Y}$ *is defined by the equality*

$$\frac{\alpha}{F^* dV_Y} = \eta|_{F^{-1}(y)}, \;\; \forall \eta \;\text{ such that }\; \eta \wedge F^* dV_Y = \alpha.$$

Proof. We prove (4.6) first. Observe that there exists a unique, compactly supported continuous function $\varphi : X \to \mathbb{R}$ such that

$$\eta \wedge F^* dV_Y = \varphi dV_X. \tag{4.8}$$

Corollary 4.4 implies that

$$\int_X \eta \wedge F^* dV_Y = \int_X \varphi dV_X = \int_Y \left(\int_{F^{-1}(y)} \frac{\varphi}{J_F} d\mathcal{H}^k(x) \right) d\mathcal{H}^n(y)$$

$$= \int_{Y^*} \left(\int_{F^{-1}(y)} \frac{\varphi}{J_F} d\mathcal{H}^k(x) \right) d\mathcal{H}^n(y) = \int_{Y^*} \left(\int_{F^{-1}(y)} \frac{\varphi}{J_F} dV_{F^{-1}(y)} \right) dV_Y.$$

To complete the proof we need to show that if y_0 is a regular value of F, then

$$\frac{\varphi}{J_F}|_{F^{-1}(y_0)} dV_{F^{-1}(y_0)} = \eta|_{F^{-1}(y_0)}.$$

We rely on the same arguments used in the proof of Theorem 2.1. Fix $x_0 \in F^{-1}(y_0)$. We can find local coordinates $y^1, \dots, y^k$ near y_0 in Y and coordinates $(x^1, \dots, x^k, x^{k+1}, \dots, x^{k+n})$ near x_0 in X such that in these coordinates F is the linear projection

$$y^j = x^j, \;\; j = 1, \dots, k.$$

We write

$$dx' = dx^{k+1} \wedge \cdots \wedge dx^{k+n}, \;\; dx'' = dx^1 \wedge \cdots \wedge dx^k, \;\; dy = dy^1 \wedge \cdots \wedge dy^k.$$

We assume that the coordinates are ordered so that

$$dV_X = \rho_X dx' \wedge dx'', \;\; dV_Y = \rho_Y dy, \;\; dV_{F^{-1}(y_0)} = \rho_F dx'.$$

[2] Since F is smooth $Y \setminus Y^*$ is negligible by Sard's theorem.

As in the proof of Theorem 2.1 we have

$$J_F = \frac{\rho_Y}{\rho_F}\rho_X.$$

In the coordinates (x', x'') we can write

$$\eta = \eta' dx' + \text{other terms}$$

where $\eta' = \eta'(x', x'')$ is a locally defined C^1-function. Note that

$$\eta|_{F^{-1}(y_0)} = \eta' dx'.$$

We deduce that

$$\eta \wedge F^* dV_Y = \eta \wedge (\rho_Y dx'') = \eta' \rho_Y dx' \wedge dx'' = \frac{\eta' \rho_Y}{\rho_X} dV_X.$$

Hence, in the coordinates (x', x'') we have

$$\varphi = \frac{\eta' \rho_Y}{\rho_X}.$$

We conclude that

$$\frac{\varphi}{J_F} dV_{F^{-1}(y_0)} = \frac{\varphi}{J_F}\rho_F dx' = \frac{\eta' \rho_Y \rho_F}{\rho_X J_F} dx' = \eta' dx' = \eta|_{F^{-1}(y_0)}.$$

Observe that (4.7) follows from (4.6). With y_0 a regular value of F as before and $x_0 \in F^{-1}(y_0)$, we write α locally near x_0 as a product

$$\alpha = \eta \wedge F^* dV_Y.$$

The form η is not unique, but its restriction to $F^{-1}(y_0)$ is. Then, by definition,

$$\eta|_{F^{-1}(y_0)} = \frac{\alpha}{F^* dV_Y}.$$

□

5. The co-area formula for Lipschitz maps between differentiable manifolds

To formulate the most general version of the co-area formula for Riemannian manifolds we need to recall a few facts about Lipschitz maps between locally Euclidean sets.

Theorem 5.1 (Rademacher)**.** *Suppose $U_k \subset \mathbb{R}^{n_k}$, $k = 0, 1$ are open sets and $F : U_0 \to U_1$ is a Lipschitz map. Then the map F is almost everywhere differentiable and the differential is a measurable map $U_0 \to \mathrm{Hom}(\mathbb{R}^{n_0}, \mathbb{R}^{n_1})$. Moreover, for any $\varepsilon > 0$ there exists a C^1 map $F_\varepsilon : U_0 \to \mathbb{R}^{n_1}$ such that*

$$\mathrm{vol}\Big(\big\{ x \in U_0;\ \ F(x) \neq F_\varepsilon(x) \,\big\}\Big) + \mathrm{vol}\Big(\big\{ x \in U_0;\ \ DF(x) \neq DF_\varepsilon(x) \,\big\}\Big) < \varepsilon \qquad \square$$

For a proof we refer to [3, §5.1].

Theorem 5.2 (Extension theorem)**.** *Suppose that $S \subset \mathbb{R}^n$ is a closed subset and $F : S \to \mathbb{R}$ is a Lipschitz function. Then f admits an extension to a Lipschitz function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ that has the same Lipschitz constant as f.* □

For a proof we refer to [3, Thm. 5.1.12].

Theorem 5.3 (The co-area formula: version 3)**.** *Suppose X and Y are C^1 Riemann manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : M \to N$ is a map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}_X^{n+k}$-measurable subset $A \subset X$ we have

$$\underbrace{\int_A J_F(x) d\mathcal{H}_X^{n+k}(x)}_{=:I(A)} = \underbrace{\int_Y \mathcal{H}_M^n\big(\, A \cap F^{-1}(y)\,\big) d\mathcal{H}_Y^k(y)}_{=:J(A)}. \tag{5.1}$$

Proof. Clearly, it suffices to prove the theorem for sets A with the following property: A is contained in a coordinate neighborhood $U_0 \subset X$ and $F(U_0)$ is contained in a coordinate neighborhood $U_1 \subset Y$ such that U_0 is bi-Lipschitz homeomorphic to a bounded open subset in $\mathbb{R}^{n+k}$ and U_1 is bi-Lipschitz homeomorphic to a bounded open set in $\mathbb{R}^k$. For any $\varepsilon > 0$ we can find a compact subset $C_\varepsilon \subset U_0$ and a C^1-map $F_\varepsilon : U_0 \to \mathbb{R}^k$ such that

$$\mathcal{H}_X^{n+k}(U_0 \setminus C_\varepsilon) < \varepsilon, \;\; F|_{C_\varepsilon} = F_\varepsilon|_{C_\varepsilon}, \;\; J_F|_{C_\varepsilon} = J_{F_\varepsilon}|_{C_\varepsilon}.$$

Then

$$I(A) - J(A) = I(A \cap C_\varepsilon) - J(A \cap C_\varepsilon) + I(A \setminus C_\varepsilon) - J(A \setminus C_\varepsilon).$$

The monotone convergence theorem implies that

$$\lim_{\varepsilon \searrow 0} I(A \setminus C_\varepsilon) = 0$$

while the Eilenberg inequality implies that

$$\lim_{\varepsilon \searrow 0} J(A \setminus C_\varepsilon) = 0.$$

On the other hand, there exists an open neighborhood V_0 of C_ε in U_0 such that $F_\varepsilon(V_0) \subset U_1$. Applying Theorem 4.1 to the C^1-map $F_\varepsilon : V_0 \to U_1$ we deduce that

$$I(C_\varepsilon) = \int_{C_\varepsilon} J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_{C_\varepsilon} J_{F_\varepsilon}(x) d\mathcal{H}_X^{n+k}(x)$$

$$\stackrel{(4.1)}{=} \int_Y \mathcal{H}_M^n\big(\, C_\varepsilon \cap F_\varepsilon^{-1}(y)\,\big) d\mathcal{H}_Y^k(y) = \int_Y \mathcal{H}_M^n\big(\, C_\varepsilon \cap F^{-1}(y)\,\big) d\mathcal{H}_Y^k(y) = J(C_\varepsilon).$$

□

Corollary 5.4. *Suppose X and Y are C^1 Riemann manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : M \to N$ is a map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}_X^{n+k}$-measurable function $\varphi : X \to \mathbb{R}$ we have

$$\int_X \varphi(x) J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_Y \mathcal{H}_M^n \left(\int_{F^{-1}(y)} \varphi(x) d\mathcal{H}^n(x) \right) d\mathcal{H}_Y^k(y), \tag{5.2}$$

as soon as either side of the equality is finite.

Proof. Note that (5.2) is linear in φ and, by Theorem 5.3, it is true for $\varphi = 1_A$. Thus it is true for linear combinations

$$\varphi = \sum_i c_i 1_{A_i}$$

The Monotone Convergence Theorem implies that it is true if φ is nonnegative. For a general φ we observe that $\varphi = \varphi^+ - \varphi^-$ and the formula is true for $\varphi^\pm$. □

Corollary 5.5 (Area formula)**.** *Let X, Y be two n-dimensional C^1-manifolds equipped with C^0-Riemann metrics and $F : X \to Y$ a Lipschitz map. Then*

$$\int_Y \# F^{-1}(y) d\mathcal{H}^n(y) = \int_X J_F(x) d\mathcal{H}^n(x). \tag{5.3}$$

□

6. The co-area formula for Lipschitz maps between rectifiable sets

A set $X \subset \mathbb{R}^L$ is said to be *countably m-rectifiable* if it is $\mathcal{H}^m$-measurable and

$$X \subset X_0 \cup \left(\bigcup_{j \geq 1} F_j(\mathbb{R}^m) \right),$$

where

- $\mathcal{H}^m(X_0) = 0$;
- the functions $F_j : \mathbb{R}^m \to \mathbb{R}^L$ are Lipschitz, $\forall j \geq 1$.

We have the following result, [3, §5.4].

Proposition 6.1. *Suppose that $X \subset \mathbb{R}^L$ is $\mathcal{H}^m$-measurable and countably m-rectifiable. Then*

$$X = \bigsqcup_{j=0}^{\infty} X_j,$$

where

- *$\mathcal{H}^m(X_0) = 0$;*
- *$X_i \cap X_j = \emptyset$ if $i \neq j$;*
- *for $j \geq 1$ there exists an m-dimensional C^1-submanifold $M_j \subset \mathbb{R}^L$ such that $X_j \subset M_j$.*

Definition 6.2. If X is a $\mathcal{H}^m$-measurable subset of $\mathbb{R}^L$, then we say that an m-dimensional vector subspace $W \subset \mathbb{R}^L$ is the approximate tangent space for X at $x \in \mathbb{R}^L$ if

$$\lim_{r \searrow 0} \int_{r^{-1}(X-y)} f(y) d\mathcal{H}^m(y) = \int_W f(y) d\mathcal{H}^m(y), \;\; \forall f \in C^0_{cpt}(\mathbb{R}^L).$$

Proposition 6.3. *Suppose that $X \subset \mathbb{R}^L$ is a countably m-rectifiable set such that $\mathcal{H}^m(X \cap K) < \infty$ for any compact subset $K \subset \mathbb{R}^L$. Then there exists a subset $X_{\mathrm{sing}} \subset X$ such that*

- *$\mathcal{H}^m(X_{\mathrm{sing}}) = 0$ and*
- *for any $x \in X \setminus X_{\mathrm{sing}}$ there exist an approximate tangent space to X at x.*

Proof. We write X as in Proposition 6.1

$$X = \bigsqcup_{j=0}^{\infty} X_j$$

where X_j is contained in a C^1, m-dimensional submanifold $M_j \subset \mathbb{R}^L$, $X_i \cap X_j = \emptyset$, $\forall i \neq j$, $\mathcal{H}^m(X_0) = 0$. For $j > 0$ we denote by X_j^* the set of points $x \in X_j$ such that

$$\lim_{r \searrow 0} \frac{\mathcal{H}^m((X \setminus X_j) \cap B_r(x))}{r^m} = \lim_{r \searrow 0} \frac{\mathcal{H}^m((M_j \setminus X_j) \cap B_r(x))}{r^m} = 0.$$

By Theorem 3.2 we have $\mathcal{H}^m(X_j \setminus X_j^*) = 0$. We will show that X admits an approximate tangent space at any point $x \in X_j^*$. Indeed, suppose $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^L)$. For simplicity assume that $\operatorname{supp} f \subset B_1(0)$, and $f \geq 0$. Then using the change in variables $y = \frac{1}{r}(z - x)$

$$\begin{aligned} \int_{\frac{1}{r}(X-x)} f(y) d\mathcal{H}^m(y) &= \frac{1}{r^m} \int_X f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) \\ &= \frac{1}{r^m} \int_{B_r(x) \cap X} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) \end{aligned}$$

Now observe that

$$\begin{aligned} &\frac{1}{r^m} \left| \int_{B_r(x) \cap X} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) - \int_{B_r(x) \cap X_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) \right| \\ &\qquad \leq \sup f \frac{\mathcal{H}^m(B_r(x) \cap (X_j \setminus X)}{r^m} \to 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{r^m} \left| \int_{B_r(x) \cap X_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) - \int_{B_r(x) \cap M_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) \right| \\ &\qquad \leq \sup f \frac{\mathcal{H}^m(B_r(x) \cap (M_j \setminus X_j)}{r^m} \to 0 \end{aligned}$$

Hence

$$\lim_{r \searrow 0} \left(\int_{\frac{1}{r}(X-x)} f(y) d\mathcal{H}^m(y) - \int_{\frac{1}{r}(M_j - x)} f(y) d\mathcal{H}^m(y) \right) = 0.$$

□

Theorem 6.4 (Extension theorem)**.** *Suppose that $C \subset \mathbb{R}^{L_1}$ is a closed subset and $F : C \to \mathbb{R}^{L_2}$ is a Lipschitz function. Then F admits an extension to a Lipschitz function $\widetilde{F} : \mathbb{R}^{L_1} \to \mathbb{R}^{L_2}$ that has the same Lipschitz constant as F.*

For a proof we refer to [3, Thm. 5.1.12].

Definition 6.5. Suppose that $X \subset \mathbb{R}^{L_1}$ is $(n+k)$-rectifiable, $Y \subset \mathbb{R}^{L_2}$ is n-rectifiable and $F : X \to Y$ is a Lipschitz map. Write $X = \bigsqcup_{j=0}^{\infty} X_j$ as in Proposition 6.1, with $X_j \subset M_j$ for $j \geq 1$ (the M_j are C^1 manifolds). The *approximate differental $apDF$* of F is defined as

$$apDF(\boldsymbol{p}) = D\tilde{F}_j(\boldsymbol{p}) \qquad \text{for } \boldsymbol{p} \in X_j,$$

where $\tilde{F}_j : M_j \to \mathbb{R}^{L_2}$ is any Lipschitz extension of $F_{|X_j} : X_j \subset M_j \to \mathbb{R}^{L_2}$. It is possible to show that $apDF(\boldsymbol{p})$ is well-defined for $\mathcal{H}^{n+k}$-a.e. $\boldsymbol{p} \in S$ and $apDF(\boldsymbol{p}) : apT_{\boldsymbol{p}}S \to \mathbb{R}^{L_2}$. The *approximate Jacobian* of F is

$$apJ_F(\boldsymbol{p}) = || \bigwedge^n apDF(\boldsymbol{p}) ||_{\mathrm{Hom}(\bigwedge^n apT_{\boldsymbol{p}}X, \bigwedge^n \mathbb{R}^{L_2})} \left(= \frac{\mathcal{H}^n(apDF(B_1(0)))}{\boldsymbol{\omega}_n} \right).$$

Remark 6.6. A few comments are in order.

(i) The above definitions do not depend on the choices of the decomposition $X = \bigsqcup_{j=0}^{\infty} X_j$ and of the extensions $\tilde{F}_j$.

(ii) $apJ_F = J_F$ in case $F : M \to N$ and M, N are C^1 manifolds of dimensions $n+k$ and n.

(iii) If N is an n-dimensional manifold and $F : X \to N$, then for $\mathcal{H}^{n+k}$-a.e. point $\boldsymbol{p} \in X$
$$apD_{\boldsymbol{p}}F : apT_{\boldsymbol{p}}X \to T_{F(\boldsymbol{p})}N.$$
(Notice that this is false if $F : X \to Y$ and Y is just assumed to be an n-rectifiable set. In this case it could even happen that the approximate tangent space $apT_{F(\boldsymbol{p})}Y$ does not exist for a subset of X with positive $\mathcal{H}^{n+k}$ measure.)

□

We are almost ready to state the final version of the coarea formula, but first we need a more sophisticated version of Theorem 3.4.

Theorem 6.7 (Eilenberg inequality - general case)**.** *Suppose (X, d_X) and (Y, d_Y) are two metric spaces and $F : X \to Y$ is Lipschitz. Suppose also that all closed balls in N are compact. Then, for all $A \subset X$, $n, k \geq 0$,*
$$\int_Y^* \mathcal{H}_X^{n+k}\big(A \cap F^{-1}(y)\big) d\mathcal{H}_Y^n(y) \leq \frac{\boldsymbol{\omega}_n \boldsymbol{\omega}_k}{\boldsymbol{\omega}_{n+k}} L^n \mathcal{H}_X^{n+k}(A).$$

For a proof of this general form of the Eilenberg inequality we refer to [2, Thm. 2.10.25].

Theorem 6.8 (The co-area formula: final version)**.** *Suppose that $X \subset \mathbb{R}^{L_1}$ is $(n+k)$-rectifiable, $Y \subset \mathbb{R}^{L_2}$ is n-rectifiable and $F : X \to Y$ is a Lipschitz map. Then, for any nonnegative $\mathcal{H}_X^{n+k}$-measurable function $\varphi : X \to \mathbb{R}$ we have*[3]
$$\int_X \varphi(\boldsymbol{p}) apJ_F(\boldsymbol{p}) d\mathcal{H}_X^{n+k}(\boldsymbol{p}) = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} \varphi(\boldsymbol{p}) d\mathcal{H}_X^k(\boldsymbol{p}) \right) d\mathcal{H}_Y^n(\boldsymbol{q}). \tag{6.1}$$

Proof. Let $X = \bigsqcup_{j=0}^{\infty} X_j$ and $Y = \bigsqcup_{i=0}^{\infty} Y_i$ as in Proposition 6.1, with $X_j \subset M_j$ and $Y_i \subset N_i$ for $i, j \geq 1$ (M_j and N_i manifolds of class C^1). Set $X_j^i = X_j \cap F^{-1}(Y_i)$, so that $X = \bigsqcup_{ij=0}^{\infty} X_j^i$.
(1) First of all, consider X_0. In this case, an application of Theorem 6.7 gives
$$\int_X (\varphi 1_{X_0})(\boldsymbol{p}) apJ_F(\boldsymbol{p}) d\mathcal{H}_X^{n+k}(\boldsymbol{p}) = 0 = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} (\varphi 1_{X_0})(\boldsymbol{p}) d\mathcal{H}_X^k(\boldsymbol{p}) \right) d\mathcal{H}_Y^n(\boldsymbol{q}).$$
(2) Second, consider X_j^0 with $j \geq 1$. Note that for all n-dimensional planes $\pi \subset \mathbb{R}^{L_2}$ there is a coordinate projection $\pi_{i_1,\ldots,i_n} : \pi \to \mathbb{R}^n$, $(x^1, \ldots, x^{L_2}) \mapsto (x^{i_1}, \ldots, x^{i_n})$ with the property that $\pi_{i_1,\ldots,i_n}^{-1}$ exists and is Lipschitz with constant $L \leq C(L_2, n)$, where $C(L_2, n)$ is a constant depending only on L_2 and n. Considering $\pi = apT_{\boldsymbol{p}}S$, this implies
$$apJ_F \leq C(L_2, n) \sum_{i_1 < \ldots < i_n} apJ_{\pi_{i_1,\ldots,i_n} \circ F}.$$

[3] Implicit in the statement of (6.1) is the fact that the various integrands are measurable with respect to the appropriate measures.

Therefore, from Theorem 5.3 applied to $\pi_{i_1,\dots,i_n}\circ F: M_j\to\mathbb{R}^n$, we get

$$\int_X (\varphi 1_{X_j^0})(\boldsymbol{p}) apJ_F(\boldsymbol{p}) d\mathcal{H}_X^{n+k}(\boldsymbol{p}) \leq C(L_2,n)\sum_{i_1<\dots<i_n}\int_{M_j}(\varphi 1_{X_j^0})(\boldsymbol{p}) apJ_{\pi_{i_1,\dots,i_n}\circ F}(\boldsymbol{p}) d\mathcal{H}_{M_j}^{n+k}(\boldsymbol{p})$$

$$= C(L_2,n)\sum_{i_1<\dots<i_n}\int_{\mathbb{R}^n}\left(\int_{(\pi_{i_1,\dots,i_n}\circ F)^{-1}(\boldsymbol{q})}(\varphi 1_{X_j^0})(\boldsymbol{p}) d\mathcal{H}_{M_j}^{k}(\boldsymbol{p})\right) d\mathcal{H}_{\mathbb{R}^n}^{n}(\boldsymbol{q})$$

$$= 0 = \int_Y\left(\int_{F^{-1}(\boldsymbol{q})}(\varphi 1_{X_j^0})(\boldsymbol{p}) d\mathcal{H}_X^k(\boldsymbol{p})\right) d\mathcal{H}_Y^n(\boldsymbol{q}),$$

where the last two equalities hold because $\mathcal{H}_{\mathbb{R}^n}^n(\pi_{i_1,\dots,i_n}(F(X^0))) = \mathcal{H}_Y^n(F(X^0)) = 0$.
(3) Now we deal with X_j^i, $i,j\geq 1$. We start by exending $F_{|X_j^i}: S_j^i\subset M_j\to Y_i\subset N_i$ to a Lipschitz map $\tilde{F}_j^i: M_j\to N_i$. We want to prove

$$\int_X (\varphi 1_{X_j^i})(\boldsymbol{p}) apJ_F(\boldsymbol{p}) d\mathcal{H}_X^{n+k}(\boldsymbol{p}) = \int_Y\left(\int_{F^{-1}(\boldsymbol{q})}(\varphi 1_{X_j^i})(\boldsymbol{p}) d\mathcal{H}_X^k(\boldsymbol{p})\right) d\mathcal{H}_Y^n(\boldsymbol{q}).$$

But this is the same thing as

$$\int_{M_j}(\varphi 1_{X_j^i})(\boldsymbol{p}) J_{\tilde{F}_j^i}(\boldsymbol{p}) d\mathcal{H}_{M_j}^{n+k}(\boldsymbol{p}) = \int_{N_i}\left(\int_{M_j\cap\tilde{F}^{-1}(\boldsymbol{q})}(\varphi 1_{X_j^i})(\boldsymbol{p}) d\mathcal{H}_{M_j}^k(\boldsymbol{p})\right) d\mathcal{H}_{N_i}^n(\boldsymbol{q}),$$

and this last equality is just Theorem 5.3.
Since the integrals are countable additive, we can combine (1), (2) and (3) together and conclude the proof. □

Corollary 6.9 (Area formula)**.** *Let $X\subset\mathbb{R}^{L_1}$ be m-rectifiable and $Y\subset\mathbb{R}^{L_2}$ be n-rectifiable, with $m\leq n$. Let $F: X\to Y$ be a Lipschitz map. Then*

$$\int_X \varphi(\boldsymbol{p}) apJ_F(\boldsymbol{p}) d\mathcal{H}_X^m(\boldsymbol{p}) = \int_Y\left(\int_{F^{-1}(\boldsymbol{q})}\varphi(\boldsymbol{p}) d\mathcal{H}_Y^0(\boldsymbol{p})\right) d\mathcal{H}_Y^m(\boldsymbol{q}) \tag{6.2}$$

for any nonnegative $\mathcal{H}_X^m$-measurable $\varphi: X\to\mathbb{R}$.

Proof. It suffices to apply 6.8 with $F(X)$ in place of Y. Indeed, the integral at the right hand side of 6.2 is in fact an integral over $F(X)\subset Y$ (the integrand being null on $Y\setminus F(X)$), and $F(X)$ is an m-rectifiable set. □

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