TOWARDS SIMPLICIAL CHERN-SIMONS THEORY I

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ABSTRACT. We consider the problem of discretizing topological field theories of Chern-Simons type. Our main instruments are Batalin-Vilkovysky formalism and standard Feynman diagram technique for functional integral in QFT. We solve the 1-dimensional case of the problem completely and check that the result does not exhibit nontrivial renormalization. Then we face the 2-dimensional case of discretizing theory on a triangle. We use Dupont construction for choosing the gauge and find the action of the discretized theory in lowest nontrivial order (containing higher classical and quantum operations). Next we check (by direct calculations, in lowest nontrivial order) that this action does possess nontrivial topological (BV-exact) renormalization flow for quantum operations under barycentric gluing, while classical operations are recovered.

We consider the problem of discretizing topological field theories of Chern-Simons type. Our main instruments are Batalin-Vilkovysky formalism and standard Feynman diagram technique for functional integral in QFT. We solve the 1-dimensional case of the problem completely and check that the result does not exhibit nontrivial renormalization. Then we face the 2-dimensional case of discretizing theory on a triangle. We use Dupont construction for choosing the gauge and find the action of the discretized theory in lowest nontrivial order (containing higher classical and quantum operations). Next we check (by direct calculations, in lowest nontrivial order) that this action does possess nontrivial topological (BV-exact) renormalization flow for quantum operations under barycentric gluing, while classical operations are recovered. Thus we partially (and completely for dimension 1) prove that gluing the discretized theory on a simplicial complex from theories on elements of triangulation is a consistent operation.

This work is part of program of A. Losev. We are very grateful to A. Losev for ideas and inspiration and to E. Getzler for we borrowed the construction of Dupont gauge from his work [1]. The results (different from ours) for higher operations on the segment were obtained earlier by A.Kozak (in letter to A. Losev) in somewhat different setting.

1. General construction

Consider a topological field theory with action

$$S = \sum_i (p_i, v_i(q)) \tag{1}$$

Here $v(q)$ is a formal vector depending on fields q that live in a (super) dGA V, $d : V \to V$ is the differential. Anti-fields (or momenta) p live in dual space $V^*[+1]$ (with flipped parity); $(\bullet, \bullet)$ means canonical pairing. We suppose that the master equation

$$\sum_i \frac{\partial^2}{\partial p_i \partial q_i} \exp(\frac{S}{\hbar}) = 0 \tag{2}$$

holds. Here index i runs over all degrees of freedom. We will consider the specific case of BF-theory (reduced version of Chern-Simons theory) when V is the algebra of differential forms on some space and $v(q) = dq + q^2$. Here master equation is equivalent to $d^2 = 0$ together with Leibnitz rule and associativity of multiplication in V.

Suppose some smaller complex V' is embedded into V as $\iota : V' \hookrightarrow V$ and $V = \iota(V') \oplus V''$. We require V'' to be an acyclic subcomplex. We call V' the space of infrared fields and V'' the space of ultraviolet fields. Thus every field $q \in V$ is split into IR- and UV-parts and differential preserves the splitting $d : V' \to V',\ d : V'' \to V''$. We denote the projectors on V' and V'' by $\mathcal{P}_{V'}$ and $\mathcal{P}_{V''}$ respectively.

We are now going to integrate out UV-degrees of freedom to define an effective action, depending on IR-fields only:

$$\exp(\frac{1}{\hbar}S_{eff}(p^{IR}, q^{IR}, \hbar)) = \int_{\mathcal{L}} \mathcal{D}p^{UV}\mathcal{D}q^{UV} \exp(\frac{1}{\hbar}S(\iota(p^{IR}) + p^{UV}, \iota(q^{IR}) + q^{UV})) \tag{3}$$

To make the quadratic form (p^{UV}, q^{UV}) in S non-degenerate we need to "fix the gauge" that is, to restrict the integration to the Lagrange submanifold

$$\mathcal{L} : Kq^{UV} = 0,\ p^{UV}K = 0 \tag{4}$$

where $K : V \to V$ is subject to the following relation:

$$d \circ K + K \circ d = \mathcal{P}_{V''} \tag{5}$$

The purpose of K is to invert d on V'' (all the cohomologies of d lying in V').

The resulting effective action also satisfies the master equation

$$\sum_j \frac{\partial^2}{\partial p_j^{IR} \partial q_j^{IR}} \exp(\frac{1}{\hbar}S_{eff}(p^{IR}, q^{IR}, \hbar)) = 0 \tag{6}$$

For S linear in anti-fields the effective action consists of the tree and one-loop parts only:

$$S_{eff}(p^{IR}, q^{IR}, \hbar) = S^0_{eff}(p^{IR}, q^{IR}) + \hbar\, S^1_{eff}(p^{IR}, q^{IR}) \tag{7}$$

The master equation (6) is then naturally split into the "classical" part

$$\{S^0_{eff}, S^0_{eff}\}_{BV} = \sum_j \frac{\partial S^0_{eff}}{\partial p_j^{IR}} \frac{\partial S^0_{eff}}{\partial q_j^{IR}} = 0 \tag{8}$$

and "quantum" part

$$\Delta_{BV} S^0_{eff} + \{S^0_{eff}, S^1_{eff}\}_{BV} = \sum_j \left(\frac{\partial^2 S^0_{eff}}{\partial p_j^{IR} \partial q_j^{IR}} + \frac{\partial S^0_{eff}}{\partial p_j^{IR}} \frac{\partial S^1_{eff}}{\partial q_j^{IR}} \right) = 0 \tag{9}$$

In the case of BF-theory, the action S is the generating function for algebraic operations on fields, differential and multiplication. The resulting S_{eff} is then interpreted as the generating function for a set of algebraic operations on IR-fields. The tree part of effective action generates the L_∞ structure on V', while the loop part generates some "quantum operations". Equation (8) is equivalent to the set of quadratic relations on L_∞ algebra and (9) intertwines the classical and quantum operations.

So this general construction of inducing algebraic structure on a subcomplex by means of the functional integral can be interpreted in two ways. The first is "physical": we are constructing the effective action for topological theory on infrared fields by integrating out ultraviolet fields. The resulting effective action allows to calculate correlation functions of the theory just by doing finite-dimensional integrals. The master equation may be regarded as an algebraic way to express locality of the action. Another, "mathematical" point of view is that we are inducing quantum homotopy algebra structure on a subcomplex of dGA.

The fact that operation of multiplication of differential forms has two different arguments q, while the action depends only on one field, is taken into account by passing to matrix valued differential forms.

2. Segment

2.1. **Definitions.** Let V be the space of matrix valued ($gl(N)$-valued, to be more specific, for some N) differential forms on a segment with de Rham differential $d : V \to V$ and wedge product $\wedge : V \otimes V \to V$. Let us denote 0-forms by $f(t)$ and 1-forms by $\omega(t)$. The corresponding anti fields are denoted p^0 and p^1 respectively. We may think of p^0 as 1-forms and of p^1 as 0-forms. Then the canonical pairing can be interpreted as

$$(p^0, f) = \mathrm{Tr} \int_0^1 p^0(t)f(t), \ (p^1, \omega) = \mathrm{Tr} \int_0^1 p^1(t)\omega(t) \tag{10}$$

We assume that parities of the fields (apart from being 0- or 1-forms) are: negative (odd) for f and p^1, positive (even) for ω and p^0.

The master invariant action on $V \oplus V^*[+1]$ takes the following form:

$$S = (p^1, df) + (p^0, f \wedge f) + (p^1, f \wedge \omega + \omega \wedge f) \tag{11}$$

Our quest now is to induce an effective action on the subcomplex V' of functions that naturally live on the end points of the segment and 1-forms living on the "bulk". We propose the following embedding ι of infrared forms in V:

$$f^{IR}(t) = f_0(1-t) + f_1 t, \ \ \omega^{IR}(t) = C dt \tag{12}$$

We call IR-forms both the elements of V' and of $\iota(V')$. So f^{IR} are just linear functions and ω^{IR} are constant 1-forms. Obviously, the space V' thus constructed is closed under action of d and contains all its cohomologies (constant functions). Ultraviolet forms are defined as those with

$$f^{UV}(0) = f^{UV}(1) = 0, \ \int_0^1 \omega^{UV}(t) = 0 \tag{13}$$

Having defined V' and V'', we know the projectors

$$\mathcal{P}_{V'} : f(t) \mapsto f(0)(1-t) + f(1)t, \ \omega(t) \mapsto \left(\int_0^1 \omega(t)\right) dt \tag{14}$$
$$\mathcal{P}_{V''} = 1 - \mathcal{P}_{V'}$$

The dual splitting of anti-fields is defined by $(p^{UV}, q^{IR}) = (p^{IR}, q^{UV}) = 0$, so

$$p^{IR\,0}(t) = p^0_0 \delta(t) + p^0_1 \delta(1-t), \ \ p^{IR\,1}(t) = p^{IR\,1} = const \tag{15}$$

and ultraviolet anti-fields are defined by

$$\int_0^1 p^{UV\,0}(t) = \int_0^1 t\, p^{UV\,0}(t) = 0, \ \int_0^1 p^{UV\,1}(t)dt = 0 \tag{16}$$

We see that under these definitions $p^{IR\,0}$ are precisely the 0-chains with support on the end points of the segment, while $p^{IR\,1}$ is a 1-chain supported on the "bulk".

Operator K inverting d on V'' is constructed uniquely

$$K : f^{UV} \mapsto 0, \ \ \omega^{UV} \mapsto \int_0^t \omega^{UV}(t') \tag{17}$$

To define K on the whole space V, we first project a form on V'' and then use d^{-1} there:

$$K : f \mapsto 0, \ \ \omega \mapsto \int_0^t \omega(t') - t \int_0^1 \omega(t') \tag{18}$$

Now the Lagrange submanifold $\mathcal{L}$, defined by (4) is

$$\mathcal{L} : \omega^{UV}(t) = 0, \ p^{UV\,0}(t) = 0 \tag{19}$$

2.2. **Calculations.** Having defined all the necessary objects, we may construct the effective action on V':

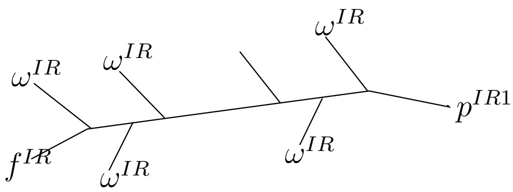
$$\exp(\frac{1}{\hbar}S_{eff}(p^{IR0},p^{IR1},f^{IR},\omega^{IR};\hbar)) = \int \mathcal{D}p^{UV1}\,\mathcal{D}f^{UV}\exp(\frac{1}{\hbar}S(p^{IR0},p^{IR1}+p^{UV1},f^{IR}+f^{UV},\omega^{IR})) \tag{20}$$

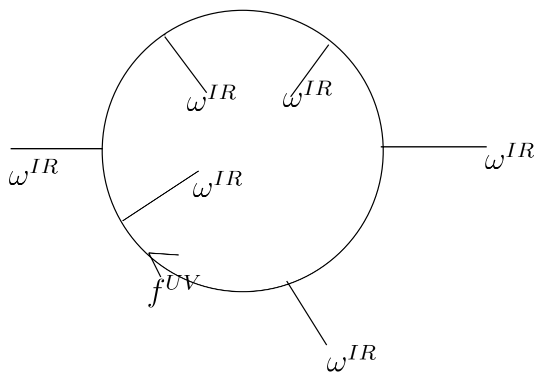
We calculate (20) perturbatively, regarding (p^{UV1},f^{UV}) as the Gaussian part of the action. Let us write down the action in the right hand side of (20) explicitly:

(21)
$$S|_{\mathcal{L}} = (p^{IR1},f_1-f_0)+(p_0^0,f_0f_0)+(p_1^0,f_1f_1)+(p^{IR1},\frac{1}{2}(f_0+f_1)\omega^{IR}+\frac{1}{2}\omega^{IR}(f_0+f_1))+ \\ +(p^{IR1},f^{UV}\omega^{IR}+\omega^{IR}f^{UV})+(p^{UV1},f^{IR}\omega^{IR}+\omega^{IR}f^{IR})+ \\ +(p^{UV1},df^{UV})+(p^{UV1},f^{UV}\omega^{IR}+\omega^{IR}f^{UV})$$

Let us denote the part depending on IR-fields only here (the first line in (21)) by $S_{IR}=S(p^{IR0},p^{IR1},f^{IR},\omega^{IR})$. The action (21) gives us K as the propagator and a bunch of vertices that basically say that a function (UV or IR one) can be multiplied by ω^{IR} from either side to produce a 1-form. The anti-field p^{UV1} serves as conjugate to f^{UV} here.

There are only two kinds of Feynman diagrams in this theory: "branches" and "loops":





Let us denote the branch with m ω^{IR} ends pointing up and n pointing down by $\mathrm{branch}_{m,n}$ (we draw f^{IR} end on the left and p^{IR1} on the left). The loop with m ω^{IR} ends pointing out and n pointing in will be denoted by $\mathrm{loop}_{m,n}$ (we assume that the function in the loop runs clockwise). It is evident from the signs in (21) that $\mathrm{branch}_{m,n}=(-1)^n\mathrm{branch}_{m+n,0}$. Let us now calculate $\mathrm{branch}_{n,0}$:

$$\mathrm{branch}_{2,0} = -(p^{IR1},\omega^{IR}K(\omega^{IR}f^{IR})),\cdots, \\ \mathrm{branch}_{n,0} = -(p^{IR1},\omega^{IR}\wedge(K(\omega^{IR}\wedge\bullet))^{n-1}\circ f^{IR}) \tag{22}$$

Let us introduce an operator κ acting on functions on the segment:

$$\kappa = K(dt\wedge\bullet): f(t)\mapsto \int_0^t f(t')dt' - t\int_0^1 f(t')dt' \tag{23}$$

Then

$$\mathrm{branch}_{n,0} = -(p^{IR1}, C^n(f_1 - f_0)) \times \int_0^1 dt\ \kappa^{n-1} \circ t \tag{24}$$

Iterated action of κ on t gives a sequence of Bernoulli polynomials:

$$\kappa^{n-1} \circ t = \frac{B_n(t) - B_n}{n!} \tag{25}$$

Here $B_n(t)$ are Bernoulli polynomials, defined by

$$\frac{xe^{xt}}{e^x - 1} = \sum B_n(t) \frac{x^n}{n!} \tag{26}$$

and $B_n = B_n(0)$ are Bernoulli numbers. Thus we obtain

$$\mathrm{branch}_{m,n} = (-1)^n \frac{B_{m+n}}{(m+n)!} (p^{IR1}, C^m(f_1 - f_0)C^n) \tag{27}$$

There exist $C^n_{m+n} = \frac{(m+n)!}{m!n!}$ branches of type (m, n), differing by the order of ω^{IR} ends going up and down. Therefore the tree part of effective action (apart from S_{IR}) is

$$S_{tree} = \sum_{m\geq 0, n\geq 0, m+n\geq 2} (-1)^n \frac{B_{m+n}}{m!n!} (p^{IR1}, C^m(f_1 - f_0)C^n) \tag{28}$$

The summand here gives a set of higher Massey operations on the segment.

Let us now compute the loop diagrams. The value of a loop diagram equals the trace of monodromy matrix along the loop (times the symmetry coefficient):

$$\mathrm{loop}_{m,n} = -\frac{1}{m+n} \mathrm{Tr}\ \mathcal{M}_{m,n} \tag{29}$$

(the sign here is a "minus for fermion loop") where

$$\mathcal{M}_{m,n} : f^{UV} \mapsto (K(\omega^{IR} \wedge \bullet))^m \circ (K(\bullet \wedge \omega^{IR}))^n \circ f^{UV} \tag{30}$$

We may express $\mathcal{M}_{m,n}$ through κ:

$$\mathcal{M}_{m,n} f^{UV} = (-1)^n C^m (\kappa^{m+n} \circ f^{UV}) C^n \tag{31}$$

so that

$$\mathrm{loop}_{m,n} = -(-1)^n \mathrm{Tr}\ C^m \mathrm{Tr}\ C^n \frac{1}{m+n} \mathrm{Tr}\ \kappa^{m+n} \tag{32}$$

Here the first two traces are in matrices where our forms take value, while the last trace is on the space of scalar-valued functions. A remarkable fact is that if we are computing $\mathrm{Tr}\ \kappa^n$ in monomial basis, i.e.

$$\mathrm{Tr}\ \kappa^n = \sum_{i=0}^{\infty} < t^i | \kappa^n | t^i > \tag{33}$$

we will find that only terms with $i \leq n$ are nonzero. This means that only a finite number of monomials contribute to any given loop diagram. For example,

$$\kappa^2 t^i = \frac{t^{i+2}}{(i+1)(i+2)} - \frac{t^2}{2(i+1)} + (\frac{1}{2(i+1)} - \frac{1}{(i+1)(i+2)})t \tag{34}$$

and so

$$\mathrm{Tr}\ \kappa^2 = < t|\kappa^2|t > + < t^2|\kappa^2|t^2 > = (\frac{1}{4} - \frac{1}{6}) + (-\frac{1}{6}) = -\frac{1}{12} \tag{35}$$

The result for $\mathrm{Tr}\ \kappa^n$ is

$$\mathrm{Tr}\ \kappa^n = -\frac{B_n}{n!} \tag{36}$$

for $n \geq 2$ (and the loop diagram with one end cancels out anyway). Therefore

$$\mathrm{loop}_{m,n} = (-1)^n \frac{B_{m+n}}{(m+n)(m+n)!} \mathrm{Tr}\ C^m \mathrm{Tr}\ C^n \tag{37}$$

and the loop part of effective action is

$$S_{loop} = S^1_{eff} = \sum_{m\geq 0, n\geq 0, m+n\geq 2} (-1)^n \frac{B_{m+n}}{(m+n)m!n!} \mathrm{Tr}\ C^m \mathrm{Tr}\ C^n \tag{38}$$

So we have computed all the contributions to the effective action on the segment:

$$\begin{aligned} S_{eff} &= S_{IR} + S_{tree} + \hbar S_{loop} = \\ &= (p^0_0, f^2_0) + (p^0_1, f^2_1) + (p^{IR1}, (f_1 - f_0) + (\frac{1}{2}(f_0+f_1)C - \frac{1}{2}C(f_0+f_1)))+ \\ &\quad + (p^{IR1}, \sum_{m\geq 0, n\geq 0, m+n\geq 2} (-1)^n \frac{B_{m+n}}{m!n!} C^m (f_1 - f_0) C^n)+ \\ &\quad + \hbar \sum_{m\geq 0, n\geq 0, m+n\geq 2} (-1)^n \frac{B_{m+n}}{(m+n)m!n!} \mathrm{Tr}\ C^m \mathrm{Tr}\ C^n \end{aligned} \tag{39}$$

2.3. **Forms of the result.** We can present the result for effective action on the segment in several ways. One form is the following: we can use a sort of Mellin transformation for Bernoulli numbers:

$$B_n = \frac{1}{2\pi i} \int_\gamma s^n\ \psi'(s+1)\ ds \tag{40}$$

where contour γ is a circle of infinite radius on the complex plane run counterclockwise, $\psi(s) = \frac{d}{ds} \log \Gamma(s)$. Using this representation, we obtain

$$S_{tree} = \frac{1}{2\pi i} \int_\gamma ds (\psi'(s+1) - \frac{1}{s} + \frac{1}{2s^2})(p^{IR1}, e^{sC}(f_1 - f_0)e^{-sC}) \tag{41}$$

$$S_{loop} = -\frac{1}{2\pi i} \int_\gamma ds \frac{\psi(s+1)}{s} \mathrm{Tr}\ e^{sC}\ \mathrm{Tr}\ e^{-sC} \tag{42}$$

If we evaluate these integrals as sums over poles, we will end up with something like a Fourier series for the effective action:

$$S_{tree} = \frac{1}{2}(p^{IR1}, [C, f_1 - f_0]) + \sum_{k=1}^\infty (p^{1IR}, e^{-kC}[C, f_1 - f_0]e^{kC}) \tag{43}$$

$$S_{loop} = -\sum_{k=1}^\infty \frac{1}{k} \mathrm{Tr}\ e^{-kC}\ \mathrm{Tr}\ e^{kC} \tag{44}$$

Another way to rewrite the effective action is as follows: note that the last term in (11) could be written as $-(p^1, [\omega, f]) = -(p^1, \mathrm{ad}_\omega \circ f)$, where ad_ω is the adjoint action in matrices. Then we can collect the branches and loops with equal number of ends:

$$\sum_{m=0}^{n} \mathrm{branch}_{n,n-m} = \frac{B_n}{n!}(p^{IR1}, \mathrm{ad}_C^n \circ (f_1 - f_0)) \tag{45}$$

$$\sum_{m=0}^{n} \mathrm{loop}_{n,n-m} = \frac{B_n}{n!}\mathrm{Tr}\ \mathrm{ad}_C^n \tag{46}$$

(we understand trace in the last line as a trace in $gl(N)^* \otimes gl(N)$). Then our series involving Bernoulli numbers collect into

$$S_{tree} = (p^{IR1}, \frac{1}{2}\mathrm{ad}_C \coth\frac{1}{2}\mathrm{ad}_C \circ (f_1 - f_0)) \tag{47}$$

$$S_{loop} = \mathrm{Tr}\ \log\frac{\sinh\frac{1}{2}\mathrm{ad}_C}{\frac{1}{2}\mathrm{ad}_C} \tag{48}$$

The integral for the effective action (20) we were computing is Gaussian in the case of segment. So it is no surprise that it can be evaluated exactly. We actually have chosen the long way to the result (47–48), expanding the action around (p^{UV1}, df^{UV}).

2.4. **Master equation.** We may check directly the master equation for our S_{eff}. The classical part yields

$$\begin{aligned} 0 = {} & \frac{\partial S^0_{eff}}{\partial p^0_0}\frac{\partial S^0_{eff}}{\partial f_0} + \frac{\partial S^0_{eff}}{\partial p^0_1}\frac{\partial S^0_{eff}}{\partial f_1} + \frac{\partial S^0_{eff}}{\partial p^{IR1}}\frac{\partial S^0_{eff}}{\partial C} = \\ & = (p^{IR1}, \frac{1}{4}(C(f_1 - f_0)^2 - (f_1 - f_0)^2 C) - \\ & - \sum_{m,n\geq 0}\sum_{m',n'\geq 0}\sum_{k=1}^{m}(-1)^{n+n'}\frac{\tilde{B}_{m+n}\tilde{B}_{m'+n'}}{m!n!m'!n'!}C^{k-1+m'}(f_1 - f_0)C^{n'+m-k}(f_1 - f_0)C^n + \\ & + \sum_{m,n\geq 0}\sum_{m',n'\geq 0}\sum_{l=1}^{n}(-1)^{n+n'}\frac{\tilde{B}_{m+n}\tilde{B}_{m'+n'}}{m!n!m'!n'!}C^m(f_1 - f_0)C^{l-1+m'}(f_1 - f_0)C^{n'+n-l}) \end{aligned} \tag{49}$$

We introduced here the notation $\tilde{B}_n = B_n$ for $n \neq 1$ and $\tilde{B}_1 = 0$. Collecting together monomials of same type $C^a(f_1 - f_0)C^b(f_1 - f_0)C^c$ in (49) we get a set of quadratic relations on Bernoulli numbers.

The quantum part of master equation gives

$$\begin{aligned} 0 = {} & \frac{\partial^2 S^0_{eff}}{\partial p^{IR1}\partial C} + \frac{\partial S^0_{eff}}{\partial p^{IR1}}\frac{\partial S^1_{eff}}{\partial C} = \\ & = \sum_{m,n\geq 0}\sum_{k=1}^{m}(-1)^n\frac{\tilde{B}_{m+n}}{m!n!}\mathrm{Tr}\ C^{k-1}\ \mathrm{Tr}\ C^{m+n-k}(f_1 - f_0) + \\ & + \sum_{m,n\geq 0}\sum_{l=1}^{n}(-1)^n\frac{\tilde{B}_{m+n}}{m!n!}\mathrm{Tr}\ C^{n-l}\ \mathrm{Tr}\ C^{m+l-1}(f_1 - f_0) + \\ & + 2\sum_{m,n\geq 0}(-1)^n\frac{\tilde{B}_{m+n}}{(m+n)m!n!}n\mathrm{Tr}\ C^m\ \mathrm{Tr}\ C^{n-1}(f_1 - f_0) \end{aligned} \tag{50}$$

This equation could in turn be interpreted as a set of linear relations on Bernoulli numbers.

2.5. **Gluing.** The next property we would expect from our discretized theory is the property of gluing: we build effective action on the complex of two segments with one common end point and then integrate this middle point out. We would like that the resulting action on the segment would coincide with the one we know (maybe up to BV-exact terms). Thus we are checking the renormalization group (in Wilson sense) for our results.

Consider the complex of three points 0,1,2 (those are labels, not coordinates) and two segments between them, (01) and (12). We embed infrared functions as piecewise linear continuous functions (with point 1 the only possible braking point) and infrared 1-forms as piecewise constant 1-forms. Then we proceed analogously to the case of one segment to obtain the effective action on this two-segment complex. The result is

$$(51)\quad S_{012} = (p_0^0, f_0^2) + (p_1^0, f_1^2) + (p_2^0, f_2^2) + \\ + (p_{01}^1, (f_1 - f_0) + \frac{1}{2}(f_0 + f_1)C_{01} - \frac{1}{2}C_{01}(f_0 + f_1) + \sum_{m+n\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n C_{01}^m (f_1 - f_0) C_{01}^n) + \\ + (p_{12}^1, (f_2 - f_1) + \frac{1}{2}(f_1 + f_2)C_{12} - \frac{1}{2}C_{12}(f_1 + f_2) + \sum_{m+n\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n C_{12}^m (f_2 - f_1) C_{12}^n) + \\ + \hbar \sum_{m+n\geq 2} \frac{B_{n+m}}{(n+m)n!m!}(-1)^n (\mathrm{Tr}\, C_{01}^m\, \mathrm{Tr}\, C_{01}^n + \mathrm{Tr}\, C_{12}^m\, \mathrm{Tr}\, C_{12}^n)$$

Now we need to separate fields on the (123) complex into IR- and UV-parts:

$$(52)\quad f_1 = \tilde{\alpha} f_0 + \alpha f_2 + f_{UV},\ C_{01} = \alpha C + C_{UV},\ C_{12} = \tilde{\alpha} C - C_{UV}, \\ p_1^0 = p_{UV}^0,\ p_{01}^1 = p_{IR}^1 - \frac{1}{\alpha} p_{UV}^1,\ p_{12}^1 = p_{IR}^1 + \frac{1}{\tilde{\alpha}} p_{UV}^1$$

Here $f_0, f_2, C, p_0^0, p_2^0, p_{IR}^1$ are the new infrared fields. Parameter α defines the embedding, and has the sense of length of segment (01) divided by length of segment (02); $\tilde{\alpha} = 1 - \alpha$. To pass from (012) complex to (02) complex we need to restrict S_{012} to the Lagrange submanifold $\mathcal{L} : p_{UV}^0 = 0,\ C_{UV} = 0$ and integrate over the remaining ultraviolet fields f_{UV}, p_{UV}^1:

$$(53)\qquad \exp(S_{02}/\hbar) = \int dp_{UV}^1 df_{UV} \exp(S_{012}|_{\mathcal{L}}/\hbar)$$

The restricted action is

$$(54)\quad S_{012}|_{\mathcal{L}} = (p_0^0, f_0^2) + (p_2^0, f_2^2) + (p_{IR}^1, (f_2 - f_0) + \frac{1}{2}(f_0 + f_2)C - \frac{1}{2}C(f_0 + f_2) + \\ + \sum_{n+m\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n (\alpha^{n+m+1} + \tilde{\alpha}^{n+m+1}) C^m (f_2 - f_0) C^n) + \\ + (p_{IR}^1, \frac{1}{2} f_{UV} C - \frac{1}{2} C f_{UV} + \sum_{n+m\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n (\alpha^{n+m} - \tilde{\alpha}^{n+m}) C^m f_{UV} C^n) + \\ + (p_{UV}^1, \frac{1}{2}(f_2 - f_0)C - \frac{1}{2}C(f_2 - f_0) - \sum_{n+m\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n (\alpha^{n+m} - \tilde{\alpha}^{n+m}) C^m (f_2 - f_0) C^n) + \\ + (p_{UV}^1, -\frac{1}{\alpha\tilde{\alpha}} f_{UV} - \sum_{n+m\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n (\alpha^{n+m-1} + \tilde{\alpha}^{n+m-1}) C^m f_{UV} C^n) + \\ + \hbar \sum_{n+m\geq 2} \frac{B_{n+m}}{(n+m)n!m!}(-1)^n (\alpha^{n+m} + \tilde{\alpha}^{n+m}) \mathrm{Tr}\, C^m\, \mathrm{Tr}\, C^n$$

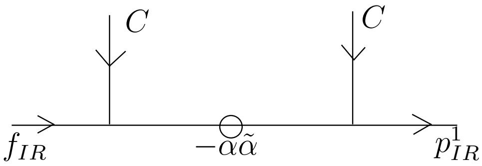
We could now calculate (53) perturbatively, considering $(p^1_{UV}, -\frac{1}{\alpha\tilde\alpha} f_{UV})$ the Gaussian part of $S_{012}|_{\mathcal{L}}$. Let us denote by $\hat{S}_{02}$ the action induced on the segment (02) directly from the continuous theory

$$\hat{S}_{02} = (p^0_0, f^2_0) + (p^0_2, f^2_2) + (p^1_{IR}, (f_2 - f_0) + \frac{1}{2}(f_0 + f_2)C - \frac{1}{2}C(f_0 + f_2) + \\ + \sum_{n+m\geq 2} \frac{B_{n+m}}{n!m!}(-1)^n C^m (f_2 - f_0) C^n) + \hbar \sum_{n+m\geq 2} \frac{B_{n+m}}{(n+m)n!m!}(-1)^n \mathrm{Tr}\, C^n\, \mathrm{Tr}\, C^m \tag{55}$$

It turns out that $S_{02} = \hat{S}_{02}$, which means the absence of renormalization. This identity may be checked order by order in C. In $\mathcal{O}(C^0)$ and $\mathcal{O}(C^1)$ it is obvious. In $\mathcal{O}(C^2)$ we must take into account terms

$$\frac{B_2}{2!}(\alpha^3 + \tilde\alpha^3)(p^1_{IR}, C^2(f_2 - f_0) - 2C(f_2 - f_0)C + (f_2 - f_0)C^2) + \\ + \hbar \frac{B_2}{2\times 2!}(\alpha^2 + \tilde\alpha^2)(\mathrm{Tr}\, C^2\, \mathrm{Tr}\, 1 - 2\mathrm{Tr}\, C\, \mathrm{Tr}\, C + \mathrm{Tr}\, 1\, \mathrm{Tr}\, C^2) \tag{56}$$

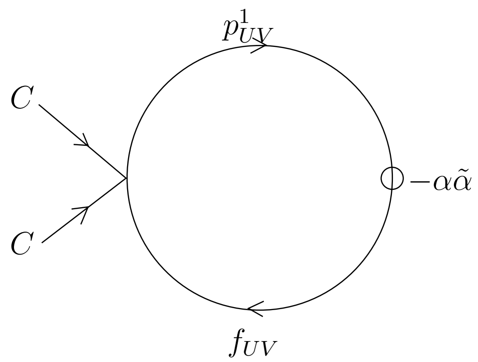
from $S_{012}|_{\mathcal{L}}$; the tree diagrams like



where C ends may point either up or down. These four diagrams together give

$$\frac{1}{4}\alpha\tilde\alpha(p^1_{IR}, C^2(f_2 - f_0) - 2C(f_2 - f_0)C + (f_2 - f_0)C^2) \tag{57}$$

At last we need loop diagrams like



where again the C ends may point either in or out. These give

$$\hbar \frac{B_2}{2!}\alpha\tilde\alpha(\mathrm{Tr}\, C^2\, \mathrm{Tr}\, 1 - 2\mathrm{Tr}\, C\, \mathrm{Tr}\, C + \mathrm{Tr}\, 1\, \mathrm{Tr}\, C^2) \tag{58}$$

Collecting all these parts together we obtain

$$\frac{B_2}{2!}(\alpha^3 + 3\alpha\tilde\alpha + \tilde\alpha^3)(p^1_{IR}, C^2(f_2 - f_0) - 2C(f_2 - f_0)C + (f_2 - f_0)C^2) + \\ + \hbar \frac{B_2}{2\times 2!}(\alpha^2 + 2\alpha\tilde\alpha + \tilde\alpha^2)(\mathrm{Tr}\, C^2\, \mathrm{Tr}\, 1 - 2\mathrm{Tr}\, C\, \mathrm{Tr}\, C + \mathrm{Tr}\, 1\, \mathrm{Tr}\, C^2) \tag{59}$$

Since $\alpha^3 + 3\alpha\tilde\alpha + \tilde\alpha^3 = \alpha^2 + 2\alpha\tilde\alpha + \tilde\alpha^2 = 1$, what we obtained is precisely the quadratic in C part of $\hat{S}_{02}$. Hence we proved the identity $S_{02} = \hat{S}_{02}$ up to order $\mathcal{O}(C^2)$. These calculations way be continued in precisely the same manner to check the absence of renormalization in higher orders in C.

Another way to calculate (53) is to avoid perturbative expansions and evaluate the (Gaussian) integral exactly using the form (47–48).

2.6. **Summary.** Let us summarize the results: we derived the effective action on segment that is a proper discretization of the 1-dimensional Chern-Simons type theory. This effective action is a generating functional for the set of higher classical and quantum operations on discretized forms. The classical operations form an L_∞ deformation of the former natural operations on continuous differential forms. The master equation for the effective action was checked directly. At last we checked that gluing two segments yields precisely the effective action on a single segment, and that means absence of renormalization for this theory. The exact solvability of the 1-dimensional problem is due to the fact that the only sensible gauge condition (the Lagrange submanifold $\mathcal{L}$, see (19)) makes the functional integral (3) Gaussian. We have no freedom in choice of operator K, and hence of $\mathcal{L}$, on the segment. Our embedding of infrared forms as linear functions and 1-forms as constant forms is not unique, but any other embedding (that forms a subcomplex, contains constant functions (cohomologies of d), and has the same dimension) differs from our embedding by a diffeomorphism of the segment. Hence our results for the 1-dimensional case seem to be universal, at least in the framework of construction described in the beginning.

3. Triangle

3.1. **Infrared and ultraviolet fields.** We now proceed to the 2-dimensional case of the problem considered in the previous section. Our fields (living in V) are now differential forms on the triangle: f is a function, ω is 1-form, Ω is a 2-form. The corresponding anti-fields are p^0, p^1 and p^2. All forms and anti-fields are again assumed to take values in $gl(N)$ for some N. The (internal) parities are 1,0,1 for f,ω,Ω and 0,1,0 for p^0, p^1, p^2. Our first task is to embed discretized forms, living on vertices, edges and face of the triangle into the space V of continuous forms on triangle as "infrared" forms $\iota : V' \hookrightarrow V$. Informally, thus we are interpolating forms into the bulk of triangle. Let the vertices of the triangle be labeled 1,2,3 (say, counterclockwise). Then edges have natural labels (12),(23),(31) and we may prescript label (123) to the bulk (we do not need it, while we consider the complex on one triangle). Then V' consists of discretized functions (f_1, f_2, f_3), 1-forms $(\omega_{12}, \omega_{23}, \omega_{31})$ and 2-forms $\Omega = \Omega_{123}$ (all these entities are constants in $gl(N)$). Hence V' is of dimension $7 = 3 + 3 + 1$ over $gl(N)$.

Let us present the nice construction for ι, borrowed from [1]. We will use homogeneous (barycentric) coordinates on the triangle: (t_1, t_2, t_3) with all $t_i \geq 0$ and $t_1 + t_2 + t_3 = 1$. Then there are defined three 1-forms dt_i subject to the relation $dt_1 + dt_2 + dt_3 = 0$. Now we introduce the elementary, or Whitney, forms as

$$\chi_{i_0 \ldots i_k} = k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \ldots \hat{dt_{i_j}} \ldots dt_{i_k} \tag{60}$$

Here the hat means exclusion. For our case this definition means

$$\begin{aligned} \chi_1 = t_1,\ \chi_2 = t_2,\ \chi_3 = t_3, \\ \chi_{12} = t_1 dt_2 - t_2 dt_1,\ \chi_{23} = t_2 dt_3 - t_3 dt_2,\ \chi_{31} = t_3 dt_1 - t_1 dt_3 \\ \chi_{123} = 2(t_1 dt_2 \wedge dt_3 + t_2 dt_3 \wedge dt_1 + t_3 dt_1 \wedge dt_2) \end{aligned} \tag{61}$$

These forms span the subcomplex of infrared forms $\iota(V') \subset V$. We define the embedding ι as

$$\begin{aligned}(62)\quad & \iota : (f_1, f_2, f_3) \mapsto f_1 t_1 + f_2 t_2 + f_3 t_3, \\ & \iota : (\omega_{12}, \omega_{23}, \omega_{31}) \mapsto \omega_{12}\chi_{12} + \omega_{23}\chi_{23} + \omega_{31}\chi_{31}, \\ & \iota : \Omega_{123} \mapsto \Omega_{123}\chi_{123}\end{aligned}$$

Thus the infrared functions are just (all) linear functions on the triangle, IR 1-forms are some special linear forms, IR 2-forms are proportional to the area form. The elementary forms have the following nice property: $\chi_{i_0 \dots i_k}$ restricted to the edge (face, etc.) $(i_0, \dots i_k)$ is constant and its integral is 1, while, being restricted to any other edge of same dimensionality it gives zero. This justifies their use for interpolating discretized forms.

We define ultraviolet fields in the following way: f^{UV} is zero on the vertices: $f_1^{UV} = f_2^{UV} = f_3^{UV} = 0$; ω^{UV} has zero integral on the edges: $\int_{(ij)} \omega^{UV} = 0$ for any edge (ij); Ω^{UV} has zero integral over the bulk: $\int_{(123)} \Omega^{UV} = 0$. Let us now construct the projectors. Let I be the integration operator $I : V \to V'$ defined as

$$\begin{aligned}(63)\quad & I : f \mapsto (f(0), f(1), f(2)), \\ & I : \omega \mapsto \left(\int_{(12)} \omega, \int_{(23)} \omega, \int_{(31)} \omega\right), \\ & I : \Omega \mapsto \int_{(123)} \Omega\end{aligned}$$

Then the projector $V \to V'$ is defined as

$$\mathcal{P}_{V'} = I \tag{64}$$

So $\mathcal{P}_{V'}$ takes a form and integrates it over the edges of proper dimension. The projector on V'' is $\mathcal{P}_{V''} = 1 - \iota \circ \mathcal{P}_{V'}$

The splitting $V = V' \oplus V''$ induces the dual splitting for anti-fields. Discretized anti-fields are just the chains of the discretized complex. Denote the space of discretized anti-fields by V'^* (which is natural, because it is dual space for V', the cochains on the discretized complex). Then V'^* consists of 0-chains (p_0^0, p_1^0, p_2^0), 1-chains $(p_{12}^1, p_{23}^1, p_{31}^1)$ and 2-chains $p^2 = p_{123}^2$. The embedding $\tilde{\iota} : V'^* \to V^*$ multiplies $p^k_{i_0 \dots i_k}$ by delta-function (delta-form, to be more precise) $\delta_{i_0 \dots i_k}$ with support on the corresponding edge $(i_0, \dots, i_k)$. We understand $p^k \in V^*$ as a $gl(N)$-valued $(2-k)$-form (with prescribed internal parity), and the pairing between fields and anti-fields is

$$(p^k, \omega^k) = \mathrm{Tr} \int_{(123)} p^k \wedge \omega^k \tag{65}$$

where ω^k is f, ω or Ω when $k = 0, 1, 2$ correspondingly.

3.2. **Dupont gauge.** Now, having defined the splitting of fields into infrared and ultraviolet parts, we face the next question, the question of gauge, or of Lagrange submanifold, or of operator K. Contrary to the 1-dimensional case, the choice of K is not unique now, and the answers will depend on it. We adopt the construction for K, suggested by Dupont and cited in [1]. We first reproduce (for the sake of convenience) the general construction, working for simplex Δ^n of any dimension n, following [1]. The next paragraph is a direct citation from [1], we only slightly changed the notation to match ours.

Given a vertex e_i of Δ^n (in our notations i runs from 1 to $n + 1$), define the dilation map

$$\phi_i : [0, 1] \times \Delta^n \to \Delta^n \tag{66}$$

by the formula

$$\phi_i(u, t_1, \dots, t_{n+1}) = (ut_1, \dots, ut_i + (1-u), \dots, ut_{n+1}) \tag{67}$$

Let $\pi : [0,1] \times \Delta^n \to \Delta^n$ be the projection on the second factor, and let $\pi_* : \Omega^*([0,1] \times \Delta^n) \to \Omega^{*-1}(\Delta^n)$ be integration over the first factor. Define operators

$$h^i_n : \Omega^*_n \to \Omega^{*-1}_n \tag{68}$$

by the formula

$$h^i_n \omega = \pi_* \phi_i^* \omega \tag{69}$$

Let $\epsilon^i_n : \Omega_n \to gl(N)$ be evaluation at the vertex e_i. Stokes's theorem implies that h^i_n is a chain homotopy between the identity and ϵ^i_n:

$$dh^i_n + h^i_n d = \mathrm{id}_n - \epsilon^i_n \tag{70}$$

The operator

$$K_n = \sum_{k=0}^{n-1} (-1)^k \sum_{0 \le i_0 < \dots < i_k \le n} \chi_{i_0 \dots i_k} h^{i_k} \dots h^{i_0} \tag{71}$$

was introduced by Dupont.

This operator $K_n : V \to V''$ is precisely the one we were. looking for. The defining property of K, that it inverts d on V'', see (5), was also proved by Dupont and cited in [1] under the name of de Rham theorem.

Now we present the specialized version for the case of triangle ($n = 2$). Since K lowers the degree of the form, $K \circ f = 0$. To define $K \circ \omega$, we decompose $\omega = \omega_{12}\chi_{12} + \omega_{23}\chi_{23} + \omega_{31}\chi_{31}$ where $\omega_{ij} = \omega_{ij}(t_1, t_2, t_3)$ are now functions on triangle. We should keep in mind that the elementary 1-forms are not independent: $t_1\chi_{23} + t_2\chi_{31} + t_3\chi_{12} = 0$. For the dilation map $\phi_i : [0,1] \times \Delta \to \Delta$ (here Δ means the triangle), pulling a point on triangle towards one the i-th vertex, we have

$$\begin{aligned} \phi_1(u)(t_1, t_2, t_3) &= (ut_1 + 1 - u, ut_2, ut_3), \\ \phi_2(u)(t_1, t_2, t_3) &= (ut_1, ut_2 + 1 - u, ut_3), \\ \phi_3(u)(t_1, t_2, t_3) &= (ut_1, ut_2, ut_3 + 1 - u) \end{aligned} \tag{72}$$

The operators h^i from 1-forms to functions are

$$\begin{aligned} h^1 &: \omega \mapsto \int_0^1 du(t_2\, \omega_{12} \circ \phi_1(u) - t_3\, \omega_{31} \circ \phi_1(u)) \\ h^2 &: \omega \mapsto \int_0^1 du(t_3\, \omega_{23} \circ \phi_2(u) - t_1\, \omega_{12} \circ \phi_2(u)) \\ h^3 &: \omega \mapsto \int_0^1 du(t_1\, \omega_{31} \circ \phi_3(u) - t_2\, \omega_{23} \circ \phi_3(u)) \end{aligned} \tag{73}$$

The purpose of h^i is to integrate ω along the segment, connecting point (t_1, t_2, t_3) and i-th vertex. Action of K on 1-forms is then

$$K : \omega \mapsto t_1 h^1\omega + t_2 h^2\omega + t_3 h^3\omega \tag{74}$$

Denote the edge opposite to the i-th vertex by $op(i)$, so that $op(1) = (23)$, $op(2) = (31)$ and $op(3) = (12)$. Operators h^i act on 2-forms $\Omega\chi_{123}$ as follows:

$$h^i(\Omega\chi_{123}) = \chi_{op(i)} 2 \int_0^1 udu\, \Omega \circ \phi_i(u) \tag{75}$$

We can now write down explicit expression for the action of K on 2-forms:

$$
\begin{aligned}
(76)\quad & K(\Omega\chi_{123}) = \\
&= (t_1 h^1 + t_2 h^2 + t_3 h^3 - \chi_{12} h^2 h^1 - \chi_{23} h^3 h^2 - \chi_{31} h^1 h^3) \circ (\Omega\chi_{123}) = \\
&= 2\chi_{12} t_3 \left(\int_0^1 udu\, \Omega \circ \phi_3(u) - \int_0^1 du \int_0^1 vdv\, \Omega \circ \phi_1(v) \circ \phi_2(u) \right) + \\
&+ 2\chi_{23} t_1 \left(\int_0^1 udu\, \Omega \circ \phi_1(u) - \int_0^1 du \int_0^1 vdv\, \Omega \circ \phi_2(v) \circ \phi_3(u) \right) + \\
&\qquad + 2\chi_{31} t_2 \left(\int_0^1 udu\, \Omega \circ \phi_2(u) - \int_0^1 du \int_0^1 vdv\, \Omega \circ \phi_3(v) \circ \phi_1(u) \right)
\end{aligned}
$$

The introduced operator K defines a Lagrange submanifold $\mathcal{L}$ by (4). It is easy to see that condition (4) admits any f^{UV} and p^{UV2} (since K is automatically zero on them), kills Ω^{UV} and p^{UV0} entirely (just like with the form of highest degree and anti-field of lowest degree on segment). So Ω and p^0 are non-dynamical fields. For ω^{UV} and p^1 condition (4) is nontrivial. The allowed 1-forms are in the image of K.

3.3. **Effective action.** The master invariant continuous action on the triangle is

$$(77)\qquad S = (p^0, f \wedge f) + (p^1, df + f \wedge \omega + \omega \wedge f) + (p^2, d\omega + \omega \wedge \omega + f \wedge \Omega + \Omega \wedge f)$$

The only way to deform this action, preserving master invariance and not introducing new "interactions" is to renormalize the fields and anti-fields in self consistent fashion:

$$(78)\qquad f \mapsto Z_f f,\ p^0 \mapsto Z_f^{-1} p^0,\ \omega \mapsto Z_\omega \omega,\ p^1 \mapsto Z_\omega^{-1} p^1,\ \Omega \mapsto Z_\Omega \Omega,\ p^2 \mapsto Z_\Omega^{-1} p^2$$

The effective action is defined by (3). It is right to think of effective action as being defined on the 14-dimensional (7 for chains and 7 for cochains) space of discretized fields and anti-fields $W \oplus W^*$. Effective action is (as in 1-dimensional case) naturally split into three parts:

$$(79)\qquad S_{eff} = S_{IR} + S_{tree} + \hbar S_{loop}$$

The first term here is $S_{IR} = S(p^{IR}, q^{IR})$ (q denotes all forms, as in introduction). Given the splitting $V = \iota(V') \oplus V''$, this object is well defined (and does not depend on the gauge). Let us write down explicit expression for S_{IR}:

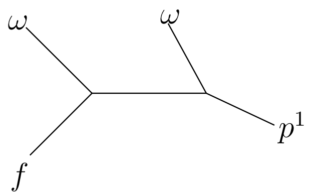
$$
\begin{aligned}
(80)\quad & S_{IR} = \\
&= (p^1_{12}, f_2 - f_1) + (p^1_{23}, f_3 - f_2) + (p^1_{31}, f_1 - f_3) + \\
&+ (p^2, \omega_{12} + \omega_{23} + \omega_{31}) + \\
&+ (p^0_1, f_1^2) + (p^0_2, f_2^2) + (p^0_3, f_3^2) + \\
&+ (p^1_{12}, [\tfrac{1}{2}(f_1 + f_2), \omega_{12}]) + (p^1_{23}, [\tfrac{1}{2}(f_2 + f_3), \omega_{23}]) + (p^1_{31}, [\tfrac{1}{2}(f_3 + f_1), \omega_{31}]) + \\
&+ (p^2, [\tfrac{1}{3}(f_1 + f_2 + f_3), \Omega]) - \\
&- (p^2, \tfrac{1}{6}([\omega_{12}, \omega_{23}] + [\omega_{23}, \omega_{31}] + [\omega_{31}, \omega_{12}]))
\end{aligned}
$$

Each line here contains definition of a discretized operation, namely of discretized versions of df, $d\omega$, $f \wedge f$, $f \wedge \omega + \omega \wedge f$, $f \wedge \Omega + \Omega \wedge f$, $\omega \wedge \omega$ (in this order). The brackets $[\bullet, \bullet]$ mean graded commutator here.

The signs behave quite tricky in the process of discretization. This is due to the fact that continuous field have "external" parity as forms and internal parity, and sum of those is always 1, so all continuous forms (that is, f, ω and Ω) in the theory are effectively odd (while all anti-fields are even for triangle). But for discretized fields, living in V' only the intrinsic parity survives. So they are effectively of alternating parity. One therefore might say that 5-th line in (80) defines rather operation $\hat{f}\hat{\wedge}\hat{\omega}-\hat{\omega}\hat{\wedge}\hat{f}$, where hat over forms means that they are discretized, that is, belong to V', and hat over wedge states that it is discretized (non associative etc.) wedge product. The last line of (80) also looks rather like $-\hat{\omega}\hat{\wedge}\hat{\omega}$. The best test for signs is master equation.

3.4. **Trees.** Contrary to the 1-dimensional case, there are many trees on triangle. For example, there exist all binary trees with many ω^{IR} ends and one p^{IR2} end. This is, of course, the sign that this theory is "interacting", and the functional integral (3) is non-Gaussian. All the trees may have many fields (infrared forms) on input, but only one anti-field (chain) on output. The trees ending on p^{IR1} naturally represent higher operations living on the edges of triangle, while trees ending on p^{IR2} represent operations living in bulk. The nice property, specific to Dupont gauge is that all higher operations living on edges coincide with higher operations on the segment (and those we know quite well). So this gauge, apart from conserving the discrete symmetry of triangle, has the wonderful property of restriction compatibility of higher operations.

Having said these general words we will present results for trees with 3 field-ends, giving trinary classical operations on triangle. The only nonzero tree, living on edges is

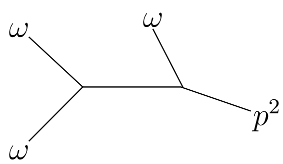


where the ω end may point either up or down. It corresponds to the following contribution to S_{tree}:

(81) $(p^{IR1}, [\omega^{IR}, K([\omega^{IR}, f^{IR}])]) =$

$$= \frac{1}{12}(p^1_{12}, [\omega_{12}, [\omega_{12}, f_2 - f_1]]) + \frac{1}{12}(p^1_{23}, [\omega_{23}, [\omega_{23}, f_3 - f_2]]) + \frac{1}{12}(p^1_{31}, [\omega_{31}, [\omega_{31}, f_1 - f_3]])$$

So, as we promised, this tree represents the operation, living on edges, that coincides with our old result for trinary operation on segment. The tree

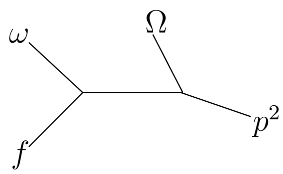


where the rightmost ω end may point up or down equals

(82) $(p^{IR2}, [\omega^{IR}, K(\omega^{IR} \wedge \omega^{IR})]) =$

$$= \frac{1}{72}(p^{IR2}, [\omega_{12}, [\omega_{12}, \omega_{12}+\omega_{23}+\omega_{31}]]+[\omega_{23}, [\omega_{23}, \omega_{12}+\omega_{23}+\omega_{31}]]+[\omega_{31}, [\omega_{31}, \omega_{12}+\omega_{23}+\omega_{31}]])$$

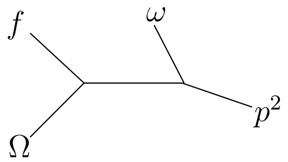
The next tree



where Ω may up or down and ω and f may interchange gives

$$(83)\quad (p^{IR2}, [K([f^{IR}, \omega^{IR}), \Omega]_+) = \\ = \frac{1}{24}(p^{IR2}, [[f_2 - f_1, \omega_{12}], \Omega]_+ + [[f_3 - f_2, \omega_{23}], \Omega]_+ + [[f_1 - f_3, \omega_{31}], \Omega]_+)$$

We explicitly wrote anticommutators here as $[\bullet, \bullet]_+$ (although the sign of commutator can anyway be seen from the parities). At last tree



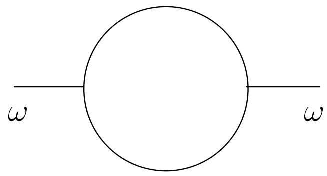
where ω may point up or down and Ω and f may interchange equals

$$(84)\quad (p^{IR2}, [\omega, K([f, \Omega]_+)]) = \\ = \frac{1}{36}(p^2, [[f_2 - f_1, \Omega]_+, \omega_{12}] + [[f_3 - f_2, \Omega]_+, \omega_{23}] + [[f_1 - f_3, \Omega]_+, \omega_{31}])$$

This concludes the list of trees with three infrared forms on input on triangle. We should remind that these results are for Dupont gauge. In other gauges the values of trees will be different and several trees that equal zero in Dupont gauge will have some values. From the experience of calculating in different gauges, it seems that Dupont gauge gives the most compact results.

3.5. **Loops.** There loop diagrams on triangle have the following structure: a single loop were an ultraviolet form runs and several trees are plugged into the loop. The trees may have any number of infrared fields on ends. Loop diagrams do not depend on infrared anti-fields (that is, as operations they have input but no output).

The loop diagrams with one end cancel (between end pointing out and end pointing in). The simplest loop diagram that gives contribution to effective action is



with ends pointing either in or out. Its value is

$$(85)\qquad \frac{1}{2}(\mathrm{Tr}\ K(\omega \wedge K(\omega \wedge \bullet_{\omega^{UV}})) - \mathrm{Tr}\ K(\omega \wedge K(\omega \wedge \bullet_{f^{UV}})))$$

The half in front is a symmetry factor, the first term is for the case when 1-form runs in the loop, the trace is a trace of operator from 1-form sector of $\mathcal{L}$ to itself. The second term is for function running in the loop, minus sign is (as for the segment) from "fermion loop".

Now we face the problem of calculating functional trace on triangle. The bad news is that while trees may be calculated in symmetric setting, in coordinates (t_1, t_2, t_3), now we need a basis in functions on triangle to calculate the diagonal elements of monodromy matrix along the loop. It is not clear how to find a symmetric basis in functions on triangle, so we chose rectangular coordinates (x, y), related to barycentric ones by $t_1 = x,\ t_2 = y,\ t_3 = 1 - x - y$. Of course, in this coordinates we lose manifest symmetry, but now we can introduce, say, monomial basis $x^a y^b$ in functions on triangle and calculate traces using it. We must hope then that final answers will be symmetric.

The first trace in (85) must be calculated on 1-forms from $\mathcal{L}$ so we rewrite it as Tr $\omega \wedge K(\omega \wedge K(\bullet_\Omega))$, a trace calculated on all 2-forms (and it is easier to introduce basis there). The calculation of (85) is rather lengthy, so we present only the results. An interesting phenomenon is that the two traces in (85) separately diverge logarithmically. We may introduce some regulator, say the degree of monomials on which we calculate the trace $x^a y^b$, $a+b \leq n$. Then the traces behave like $\sim \log n$. The wonderful thing is that in (85) the divergence cancels between cases when function or 1-form runs in the loop. This might mean that some sort of supersymmetry is at work here. After this cancelation the result becomes wonderfully symmetric:

$$\begin{aligned}(86)\quad &\frac{1}{18}\mathrm{Tr}\,(\omega_{12}^2+\omega_{23}^2+\omega_{31}^2)\ \mathrm{Tr}\,1-\frac{1}{18}((\mathrm{Tr}\,\omega_{12})^2+(\mathrm{Tr}\,\omega_{23})^2+(\mathrm{Tr}\,\omega_{31})^2)+\\ &+\frac{1}{270}\mathrm{Tr}\,((\omega_{12}+\omega_{23}+\omega_{31})^2)\ \mathrm{Tr}\,1-\frac{1}{270}(\mathrm{Tr}\,(\omega_{12}+\omega_{23}+\omega_{31}))^2\end{aligned}$$

We see that there is part $\sim$ Tr ω^2 Tr $1-(\mathrm{Tr}\,\omega)^2$ similar to what we had on the segment and part of a new kind $\sim$ Tr $(d\omega)^2$ Tr $1-(\mathrm{Tr}\ d\omega)^2$ where $d\omega$ is the discretized differential.

If we intend to glue some bigger complexes from triangles, we should ask the following question: we know where do classical operations, represented by trees, live, but where do the quantum operations live? On edges? On the bulk? To answer this question we must calculate (85) using a special basis for functions on triangle. It should consist of three sub-bases for each edge and one for bulk. For each edge we must take a basis in functions on it (say, monomials) and multiply by delta-function with support on this edge. For the bulk we must take basis in functions, vanishing on the edges, say $xy(1-x-y)x^a y^b$. Ultraviolet 1-forms automatically live in the bulk. Evaluating (85) on each of the sub-bases, we will understand what portions of the answer (86) lives on any edge, and what lives in the bulk. We present the result:

$$\begin{aligned}(87)\quad &\frac{1}{12}(\mathrm{Tr}\,\omega_{12}^2\ \mathrm{Tr}\,1-(\mathrm{Tr}\,\omega_{12})^2)+\\ &\frac{1}{12}(\mathrm{Tr}\,\omega_{23}^2\ \mathrm{Tr}\,1-(\mathrm{Tr}\,\omega_{23})^2)+\\ &\frac{1}{12}(\mathrm{Tr}\,\omega_{31}^2\ \mathrm{Tr}\,1-(\mathrm{Tr}\,\omega_{31})^2)+\\ &+\Big(-\frac{1}{36}\mathrm{Tr}\,(\omega_{12}^2+\omega_{23}^2+\omega_{31}^2)\ \mathrm{Tr}\,1+\frac{1}{36}((\mathrm{Tr}\,\omega_{12})^2+(\mathrm{Tr}\,\omega_{23})^2+(\mathrm{Tr}\,\omega_{31})^2))+\\ &+\frac{1}{270}\mathrm{Tr}\,((\omega_{12}+\omega_{23}+\omega_{31})^2)\ \mathrm{Tr}\,1-\frac{1}{270}(\mathrm{Tr}\,(\omega_{12}+\omega_{23}+\omega_{31}))^2\Big)\end{aligned}$$

The first three lines contain parts living on edges (12), (23) and (31) correspondingly, while the last two lines represent the bulk part of the loop diagram.

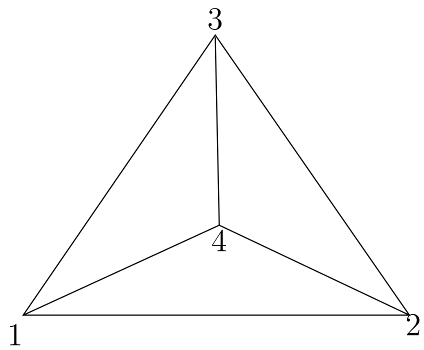
All the other loop diagrams with two end are zero. Vanishing of diagram where there is one f^{IR}-end and one Ω^{IR}-end plugged into the loop in non-evident. It was proved by direct computation.

Thus we have calculated S_{eff} on triangle up to order $\mathcal{O}(pq^3+\hbar q^2)$ where q is any IR-form, and p is any IR-chain. Note that the order "trinary classical operations+binary quantum operations" is self-consistent from the point of view of checking master equation. The master equation holds for our answers (to the order where the next operations do not interfere). A nice observation is that parts of (85) living on edges cancels in quantum part of master equation by trees living on those edges, while the bulk part cancels by trees living in bulk. Let us write down our result for S_{eff} in full form:

$$
\begin{aligned}
(88)\quad S_{eff} = S_{IR} + S_{tree} + \hbar S_{loop} = \\
= (p^1_{12}, f_2 - f_1) + (p^1_{23}, f_3 - f_2) + (p^1_{31}, f_1 - f_3) + \\
+ (p^2, \omega_{12} + \omega_{23} + \omega_{31}) + \\
+ (p^0_1, f_1^2) + (p^0_2, f_2^2) + (p^0_3, f_3^2) + \\
+ (p^1_{12}, [\frac{1}{2}(f_1 + f_2), \omega_{12}]) + (p^1_{23}, [\frac{1}{2}(f_2 + f_3), \omega_{23}]) + (p^1_{31}, [\frac{1}{2}(f_3 + f_1), \omega_{31}]) + \\
+ (p^2, [\frac{1}{3}(f_1 + f_2 + f_3), \Omega]) - \\
- (p^2, \frac{1}{6}([\omega_{12}, \omega_{23}] + [\omega_{23}, \omega_{31}] + [\omega_{31}, \omega_{12}]) + \\
+ \frac{1}{12}(p^1_{12}, [\omega_{12}, [\omega_{12}, f_2 - f_1]]) + \frac{1}{12}(p^1_{23}, [\omega_{23}, [\omega_{23}, f_3 - f_2]]) + \frac{1}{12}(p^1_{31}, [\omega_{31}, [\omega_{31}, f_1 - f_3]]) + \\
+ \frac{1}{72}(p^{IR2}, [\omega_{12}, [\omega_{12}, \omega_{12} + \omega_{23} + \omega_{31}]] + [\omega_{23}, [\omega_{23}, \omega_{12} + \omega_{23} + \omega_{31}]] + [\omega_{31}, [\omega_{31}, \omega_{12} + \omega_{23} + \omega_{31}]]) + \\
+ \frac{1}{24}(p^{IR2}, [[f_2 - f_1, \omega_{12}], \Omega]_+ + [[f_3 - f_2, \omega_{23}], \Omega]_+ + [[f_1 - f_3, \omega_{31}], \Omega]_+) + \\
+ \frac{1}{36}(p^2, [[f_2 - f_1, \Omega]_+, \omega_{12}] + [[f_3 - f_2, \Omega]_+, \omega_{23}] + [[f_1 - f_3, \Omega]_+, \omega_{31}]) + \\
+ \hbar \left(\frac{1}{18} \mathrm{Tr}\, (\omega_{12}^2 + \omega_{23}^2 + \omega_{31}^2)\, \mathrm{Tr}\, 1 - \frac{1}{18}((\mathrm{Tr}\, \omega_{12})^2 + (\mathrm{Tr}\, \omega_{23})^2 + (\mathrm{Tr}\, \omega_{31})^2) + \right. \\
\left. + \frac{1}{270} \mathrm{Tr}\, ((\omega_{12} + \omega_{23} + \omega_{31})^2)\, \mathrm{Tr}\, 1 - \frac{1}{270}(\mathrm{Tr}\, (\omega_{12} + \omega_{23} + \omega_{31}))^2 \right) + \mathcal{O}(pq^4 + \hbar q^3)
\end{aligned}
$$

Each line here represents an operation on triangle (except the quantum operation, which occupies the last two lines).

3.6. **Barycentric gluing.** We now proceed to gluing one triangle from three triangles in the spirit of calculation done earlier for the segment. The idea is to write down the discretized theory on the complex of three triangles



Denote this complex by (1234). Then we would like to "glue" the three triangles (124),(234),(314) in one (123), that is to integrate out the "ultraviolet" degrees of freedom, living on internal edges and vertex 4. Next we want to compare the result to the action (88), the result of inducing operations on forms directly from continuous theory to one triangle. If the results coincide, it would mean that there is again no renormalization for triangle. If they not, it would mean there exists nontrivial renormalization group. Then we would like it to be BV-exact.

To find the action S_{1234} on big complex (1234) we need to start from continuous theory, embed forms as piecewise-elementary with "pieces" being the smaller triangles (124),(234),(314), define operator K by Dupont construction on each of the smaller triangles and so on. The resulting action is constructed as follows: we will have the same operations as in (88), for operation living on vertices we should sum over the 4 vertices of big complex, for those living on edges we should sum over the 6 edges, for those living on bulk we will have sum over three smaller triangles. Here we will need the information (87) on how the quantum operation is distributed over edges and bulk.

The embedding of infrared/ultraviolet forms and the Lagrange submanifold are directly induced from the continuous construction. The 7-dimensional space of forms on (123) V' is embedded into 13-dimensional space of forms on (1234) V as follows:

$$f_1 = f_1,\ f_2 = f_2,\ f_3 = f_3,\ f_4 = \frac{1}{3}(f_1 + f_2 + f_3) + f_4^{UV},$$
$$\omega_{12} = \omega_{12},\ \omega_{23} = \omega_{23},\ \omega_{31} = \omega_{31},$$
$$\omega_{14} = \frac{1}{3}(\omega_{12} - \omega_{31}) + \omega_{14}^{UV},\ \omega_{24} = \frac{1}{3}(\omega_{23} - \omega_{12}) + \omega_{24}^{UV},\ \omega_{34} = \frac{1}{3}(\omega_{31} - \omega_{23}) + \omega_{34}^{UV},$$
$$\Omega_{124} = \frac{1}{3}\Omega_{123} + \Omega_{124}^{UV},\ \Omega_{234} = \frac{1}{3}\Omega_{123} + \Omega_{234}^{UV},\ \Omega_{314} = \frac{1}{3}\Omega_{123} + \Omega_{314}^{UV} \tag{89}$$

Here Ω^{UV} are assumed to satisfy $\Omega_{124}^{UV} + \Omega_{234}^{UV} + \Omega_{314}^{UV} = 0$. Fields (f_1, f_2, f_3), $(\omega_{12}, \omega_{23}, \omega_{31})$ and Ω_{123} are considered infrared. Thus we defined the splitting $V = V' \oplus V''$ of 13-dimensional space of discretized forms on big complex (1234) into 7-dimensional space of infrared forms V', isomorphic to space V' of discretized forms on small complex (123) and V'', the 6-dimensional space of ultraviolet forms.

The dual splitting for chains is:

$$p_1^0 = p_1^0,\ p_2^0 = p_2^0,\ p_3^0 = p_3^0,\ p_4^0 = p_4^{UV0},$$
$$p_{12}^1 = p_{12}^1,\ p_{23}^1 = p_{23}^1,\ p_{31}^1 = p_{31}^1,\ p_{14}^1 = p_{14}^{UV1},\ p_{24}^1 = p_{24}^{UV1},\ p_{34}^1 = p_{34}^{UV1},$$
$$p_{124}^2 = p_{123}^2 + p_{124}^{UV2},\ p_{234}^2 = p_{123}^2 + p_{234}^{UV2},\ p_{314}^2 = p_{123}^2 + p_{314}^{UV2} \tag{90}$$

where the ultraviolet 2-chains are subject to condition $p_{124}^{UV2} + p_{234}^{UV2} + p_{314}^{UV2} = 0$. Chains (p_1^0, p_2^0, p_3^0), $(p_{12}^1, p_{23}^1, p_{31}^1)$ and p_{123}^2 are considered infrared.

Discrete version of operator $K : V \to V'$, induced from continuous Dupont construction, acts as follows:

$$(K\Omega)_{14} = \frac{1}{3}(\Omega_{314} - \Omega_{124}),\ (K\Omega)_{24} = \frac{1}{3}(\Omega_{124} - \Omega_{234}),\ (K\Omega)_{34} = \frac{1}{3}(\Omega_{234} - \Omega_{314}),$$
$$(K\omega)_4 = \frac{1}{3}(\omega_{14} + \omega_{24} + \omega_{34}) \tag{91}$$

It is very easy to check for this discretized Dupont construction its main properties $K \circ d + d \circ K = \mathcal{P}_{V''}$ and $K^2 = 0$. Having defined K, we defined the Lagrange submanifold (4):

$$\mathcal{L} : \omega_{14}^{UV} + \omega_{24}^{UV} + \omega_{34}^{UV} = 0,\ \Omega_{124}^{UV} = \Omega_{234}^{UV} = \Omega_{314}^{UV} = 0,$$
$$p_4^{UV0} = 0,\ p_{14}^{UV1} = p_{24}^{UV1} = p_{34}^{UV1} = p^{UV1} \tag{92}$$

So integration space in (3) is reduced from 6+6 to 3+3 dimensions: we have to integrate over f_4^{UV}, p^{UV1}, over 2-dimensional plane $\omega_{14}^{UV} + \omega_{24}^{UV} + \omega_{34}^{UV} = 0$ and another 2-dimensional plane $p_{124}^{UV2} + p_{234}^{UV2} + p_{314}^{UV2} = 0$. Calculating

$$(93) \quad \exp(\frac{1}{\hbar} S_{123}(p^{IR}, q^{IR}; \hbar)) = \\ = \int_{\mathcal{L}} \mathcal{D}p^{UV} \mathcal{D}q^{UV} \exp(\frac{1}{\hbar} S_{1234}(p^{IR} + p^{UV}, q^{IR} + q^{UV}; \hbar))$$

is quite lengthy (it is done in the same fashion as we did gluing two segments), so we present only the result. Denote the action (88) obtained by direct inducing from continuous theory to the complex (123) by $\hat{S}_{123}$. The calculations were done as before to order $\mathcal{O}(pq^3 + \hbar q^2)$. It turned out that the tree parts of S_{123} and $\hat{S}_{123}$ coincide, while the loop parts do not. The difference between these two actions comes from the $\sim (d\omega)^2$ part of the loop action. Let us denote the coefficient of the term $\mathrm{Tr}\,(d\omega)^2\, \mathrm{Tr}\, 1 - (\mathrm{Tr}\, d\omega)^2$ in the action by g. For action (88) $g = \frac{1}{270}$. The result of barycentric gluing is that the glued action S_{123} will contain the same term with coefficient $g' = \frac{1}{3}g - \frac{2}{243}$. Thus we see that there is nontrivial renormalization group for triangle in Dupont gauge. Obviously it is BV-exact (at least in the order we can check it). We may write down the renormalization flow under successive barycentric gluing: if on small triangles the coefficient was g_0 then on the big triangle, after k successive iterations of barycentric gluing we will obtain coefficient

$$(94) \qquad g_k = (g_0 + \frac{1}{81}) \times 3^{-k} - \frac{1}{81}$$

The beta-function of this barycentric renormalization flow is linear and nonhomogeneous

$$(95) \qquad \beta(g) = \frac{\partial g}{\partial \log N_\Delta} = -\frac{1}{81} - g$$

Here $N_\Delta = 3^k$ is the number of triangles in triangulation. In the limit $k \to \infty$ we arrive at the stable point of the flow $g_\infty = -\frac{1}{81}$. Action with this coefficient of the problematic term will be automorphic, that is, it will recover itself after barycentric gluing. We may then propose the following conjecture: the effective action on triangle in Dupont gauge has nontrivial BV-exact renormalization flow under barycentric gluing for loop part (while tree part recovers itself), this flow has a stable point, where the action approaches with successive iterations of gluing, which gives automorphic action. We checked this conjecture in the lowest nontrivial order.

3.7. **Summary of results for triangle.** We calculated the effective action for discretized forms on triangle in Dupont gauge to order $\mathcal{O}(pq^3 + \hbar q^2)$. We also argued why this gauge is particulary nice: apart from being manifestly symmetric, it maintains a sort of restriction compatibility. Of particular interest here is the calculation of the loop diagram. We see there a wonderful cancelation of logarithmic divergencies between a loop with function running inside and with 1-form running inside. Which we may regard as hint on supersymmetry. Next we checked the master equation for our results by direct computation. At last, we checked the barycentric gluing of three triangles in one. We found out that (in the order we are working) the tree part of action recovers itself, while the loop part gives nontrivial topological (BV-exact) renormalization group. Latter was shown to possess a fixed point that gives automorhic action (under barycentric gluing).

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