# INTEGRATION BY PARTIAL FRACTIONS

We now turn to the problem of integrating rational functions, i.e., functions of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. We will focus on rational functions $\frac{p(x)}{q(x)}$ such that the degree of the numerator $p(x)$ is strictly less than the degree of $q(x)$, otherwise one can make use of the *long division algorithm for polynomials* to reduce $\frac{p(x)}{q(x)}$ to the sum of a polynomial and a rational function such that the numerator has degree less than the degree of the denominator. For example, given polynomials $p(x) = x^3 + 2x - 1$ and $q(x) = x^2 + x + 2$ (note that $\deg(p(x)) > \deg(q(x))$) we can use the long division algorithm for polynomials to write

$$x^3 + 2x - 1 = (x^2 + x + 2)(x - 1) + (x + 1)$$

and therefore

$$\frac{x^3 + 2x - 1}{x^2 + x + 2} = (x - 1) + \frac{x + 1}{x^2 + x + 2}.$$

Another example is the fraction $\frac{(x-1)^2}{(x+1)}$ that can be expressed as

$$\frac{(x-1)^2}{(x+1)} = \frac{(x-1)(x+1-2)}{x+1} = (x-1) - \frac{2(x-1)}{(x+1)} = (x-3) - \frac{4}{(x-1)}$$

The technique of integration by partial fractions is based on a deep theorem in algebra called *Fundamental Theorem of Algebra* which we now state

**Theorem 1.** *Let $q(x)$ be a polynomial with real coefficients, then $q(x)$ can be written as a product of two types of polynomials, namely*

(a) *Powers of linear polynomials, i.e., polynomials of the form $(x - \alpha)^k$ for $k$ a positive integer.*
(b) *Powers of irreducible polynomials of degree 2, i.e. polynomials of the form $(ax^2 + bx + c)^k$ with $k$ a positive integer and $b^2 - 4a \cdot c < 0$.*

Unfortunately we will have to take the Fundamental Theorem of Algebra and many of its important consequences on faith. Let us now review all of the cases of integration by partial cases

**Case 1.** *The denominator can be factored into linear polynomials with no multiplicity (i.e. no repeated linear factors)*

In this case we have a quotient of the form

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-\alpha_1)\cdot \ldots \cdot (x-\alpha_k)},$$

where all $\alpha_j$ for $j = 1 \dots k$, are different. One can show that one have a decomposition of the form

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-\alpha_1)} + \dots + \frac{A_k}{(x-\alpha_k)},$$

for some constants $A_1, \dots, A_k$ and the idea is to solve for $A_1, \dots, A_k$. Let us look at the following example:

$$\int \frac{(x+1)dx}{x^2-25} = \int \frac{(x+1)dx}{(x-5)(x+5)}.$$

In this case, we can write

$$\frac{x+1}{(x-5)(x+5)} = \frac{A}{(x-5)} + \frac{B}{(x+5)},$$

for some constants $A, B$. There are two basic approaches for solving for the constants $A, B$:

Method 1 (*Undetermined Coefficients*) In order to find $A, B$, we can simply combine the two fractions to get

$$\frac{x+1}{(x-5)(x+5)} = \frac{A(x+5)+B(x-5)}{(x-5)(x+5)} = \frac{(A+B)x+5A-5B}{(x-5)(x+5)}$$

and in order for the above equality to hold, one needs the coefficients of all terms in $x+1$ to equal the coefficients of $(A+B)x+5A-5B$. For example, the coefficient of $x$ in $x+1$ is 1, and this coefficient has to equal the coefficient of $x$ in $(A+B)x+5A-5B$, which is $(A+B)$. Using this observation we conclude that in order to solve for $A$ and $B$ one needs to solve the linear system

$$\begin{cases} A+B &= 1 \text{ (coefficients of } x\text{)} \\ 5A-5B &= 1 \text{ (constant terms)} \end{cases}$$

and then

$$A = 3/5,$$
$$B = 2/5.$$

With these values of $A$ and $B$ we have

$$\frac{x+1}{x^2-25} = \frac{3}{5}\frac{1}{(x-5)} + \frac{2}{5}\frac{1}{(x+5)},$$

and then

$$\int \frac{x+1}{x^2-25}dx = \frac{3}{5}\ln|x-5| + \frac{2}{5}\ln|x+5| + C.$$

Method 2 An alternative way of solving for $A$ and $B$ is to multiply through by one of the factors of the denominator, say $(x-5)$ and obtain

$$\underbrace{\frac{(x+1)}{(x+5)}}_{\text{Evaluate at } x=5} = A + \underbrace{\frac{B(x-5)}{(x+5)}}_{\text{Evaluate at } x=5}$$

and *evaluating* at $x = 5$ we obtain $A = 3/5$. Analogously we can multiply through by $(x + 5)$ to obtain

$$\frac{(x+1)}{(x-5)} = \frac{A(x+5)}{(x-5)} + B,$$

and evaluating at $x = -5$ we obtain $B = 2/5$.

**Case 2.** *The denominator can be factored into linear factors with multiplicity.*

The integrand in this case looks like

$$\frac{p(x)}{(x-\alpha_1)^{m_1} \cdot \ldots \cdot (x-\alpha_2)^{m_k}}.$$

Whenever the denominator has linear factors with multiplicities, for each factor of the form $(x - \alpha_j)^{m_j}$ in the denominator, we need to include in the decomposition of the integrand a term of the form

$$\frac{A_1}{(x-\alpha_j)} + \frac{A_2}{(x-\alpha_j)^2} \cdots + \frac{A_j}{(x-\alpha_j)^{m_j}}.$$

for some constants $A_1, \ldots, A_k$ and we again need to solve for these constants.

**Example** Compute the integral

$$\int \frac{(x^2+2x-1)dx}{x^3-2x^2}$$

Using partial fractions we have the decomposition

$$\frac{x^2+2x-1}{x^3-2x^2} = \frac{x^2+2x-1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)}$$

for some constants $A$, $B$, $C$ which we can solve for using two methods:

Method 1 We can again use the method of undetermined coefficients to solve for $A, B$ and $C$, that is, we combine the fractions on the right-hand side to obtain the equality

$$\frac{x^2+2x-1}{x^3-2x^2} = \frac{(A+C)x^2 + (B-2A)x - 2B}{x^2(x-2)},$$

which is equivalent to the system

$$\begin{cases} A + C & = 1, \\ B - 2A & = 2, \\ -2B & = -1. \end{cases}$$

The solution of the system is

$$A = -\frac{3}{4}$$
$$B = \frac{1}{2}$$
$$C = \frac{7}{4}.$$

From the decomposition

$$\frac{x^2+2x-1}{x^3-2x^2} = \frac{x^2+2x-1}{x^2(x-2)} = \frac{-3}{4x} + \frac{1}{2x^2} + \frac{7}{4(x-2)},$$

we have

$$\int \frac{x^2+2x-1}{x^3-2x^2} dx = -\frac{3}{4}\ln|x| - \frac{1}{2x} + \frac{7}{4}\ln|x-2|.$$

Method 2 Let us now multiply through by $x^2$:

$$\frac{x^2+2x-1}{x-2} = Ax + B + \frac{Cx^2}{(x-2)}$$

Evaluating at $x = 0$ we obtain $B = 1/2$. In order to solve for $A$, we observe that $\frac{Cx^2}{(x-2)}$ has $x = 0$ as a *double root* and therefore

$$\frac{d}{dx}\left(\frac{Cx^2}{(x-2)}\right)\Big|_{x=0} = 0.$$

**Remark** A differentiable function $f(x)$ has a double root at $\alpha$ if we can write $f(x) = (x-\alpha)g(x)$ where $g(x)$ is also differentiable at $x = \alpha$. Note that if $\alpha$ is a double root for $f(x)$ then

$$\frac{d}{dx}f(x)\Big|_{x=\alpha} = 0.$$

We now have

$$\frac{d}{dx}\left(\frac{x^2+2x-1}{x-2}\right)\Big|_{x=0} = A + \underbrace{\frac{d}{dx}\left(\frac{Cx^2}{(x-2)}\right)\Big|_{x=0}}_{=0},$$

and then

$$\frac{d}{dx}\left(\frac{x^2+2x-1}{(x-2)}\right)\Big|_{x=0} = \frac{x^2-4x-3}{(x-2)^2}\Big|_{x=0} = -\frac{3}{4}.$$

Finally, in order to solve for $C$ we multiply through by $(x-2)$ to obtain

$$\frac{x^2+2x-1}{x^2} = \frac{A(x-2)}{x} + \frac{B(x-2)}{x^2} + C$$

and evaluating at $x = 2$ we verify that $C = 7/4$.

**Case 3.** *The denominator is divisible by irreducible polynomials of degree* 2

If the denominator is divisible by $(ax^2 + bx + c)^k$ with $k$ a positive integer and $b^2 - 4ac < 0$, we include terns of the form

$$\frac{A_1 x + B_1}{(ax^2 + bx + c)} + \ldots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}.$$

Let us now consider several examples

Example 1 A simple example is given by the integral $\int \frac{dx}{(x^2+1)(x-1)^2}$. Using decomposition by partial fractions we obtain

$$\frac{1}{(x^2+1)(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx + D}{(x^2+1)}.$$

Before solving for $A, B, C, D$ we observe that we know how to integrate all of the above functions, in particular

$$\int \frac{Cx + D}{x^2+1} dx = C \int \frac{xdx}{x^2+1} + D \int \frac{dx}{x^2+1} = \frac{C}{2} \ln|x^2+1| + D \arctan(x).$$

**Exercise:** Find the Constants $A, B, C, D$. In this case the constants can be found using the method of undetermined coefficients.

Example 2 Compute the integral

$$\int \frac{x^4 + x^2 + 1}{x(x^2+1)^2} dx.$$

In this case we have

$$\frac{x^4 + x^2 + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{(x^2+1)} + \frac{Dx + E}{(x^2+1)^2}$$

We now explain how to compute the integral of $\frac{Dx+E}{(x^2+1)^2}$. Note that

$$\int \frac{Dxdx}{(x^2+1)^2} = \frac{D}{2} \ln(x^2+1) + \text{const.}$$

On the other hand, we can compute $\int \frac{Edx}{(x^2+1)^2}$ by means of the trigonometric substitution $x = \tan(\theta)$

$$\begin{aligned}\int \frac{Edx}{(x^2+1)^2} &= E\int \frac{\sec^2(\theta)d\theta}{\sec^4(\theta)} = E\int \cos^2(\theta)d\theta \\ &= E\int \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = E\left(\theta + \frac{1}{2}\sin(2\theta)\right) + \text{const} = \\ &= E\left(\theta + \sin(\theta)\cos(\theta)\right) + \text{const} \\ &= E\left(\arctan(x) + \frac{x}{\sqrt{1+x^2}}\right) + \text{const.}\end{aligned}$$

In general, using $x = \tan(\theta)$ we can compute any integral of the form $\int \frac{dx}{(1+x^2)^k}$ by observing that

$$\int \frac{dx}{(1+x^2)^k} = \int \frac{\sec^2(\theta)d\theta}{(\sec^2(\theta))^k} = \int \cos^{2(k-1)}(\theta)d\theta.$$

We now return to the computation of the coefficients $A, \ldots, E$. We have

$$\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{x(x^2+1)^2}$$

which is equivalent to the system

$$\begin{cases} A+B &= 1, \\ C &= 0, \\ 2A+B+D &= 1, \\ C+E &= 0, \\ A &= 1. \end{cases}$$

The solution of the system is given $A = 1$, $D = -1$ and $B = C = E = 0$. The expansion in partial fractions is now

$$\frac{x^4+x^2+1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{(x^2+1)^2}.$$

Example 3 Recall the identity

$$1 + x + x^2 = \frac{x^3-1}{x-1},$$

which comes from adding the first three terms of the geometric progression $1, x, x^2, \ldots x^n, \ldots$. Let us now compute

$$\int \frac{dx}{(x^3-1)^2} = \int \frac{dx}{(x-1)^2(x^2+x+1)^2}dx$$

We expand the integrand into partial fractions

$$\frac{1}{(x-1)^2(1+x+x^2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(1+x+x^2)} + \frac{Ex+F}{(1+x+x^2)^2}, \tag{1}$$

and in order to compute integrals of the form $\int \frac{dx}{(1+x+x^2)^k}$ and of the form $\int \frac{xdx}{(1+x+x^2)^k}$, we can complete a square

$$x^2+x+1 = (x+1/2)^2 + \frac{3}{4}$$

and with the $u$-substitution $u = x + 1/2$ we have $x^2+x+1 = u^2+\frac{3}{4}$ so that

$$\int \frac{dx}{(1+x+x^2)^k} = \int \frac{du}{\left(u^2+\frac{3}{4}\right)^k}$$

and this integral can be solved by using the substitution $u = \frac{\sqrt{3}}{2}\tan(\theta)$. We also have

$$\int \frac{xdx}{(1+x+x^2)^k} = \int \frac{udu}{\left(u^2+\frac{3}{4}\right)^k} - \frac{1}{2}\int \frac{du}{(u^2+\frac{3}{4})^k},$$

and $\int \frac{udu}{(u^2+3/4)^k}$ can be computed using a $u$-substitution, in fact

$$\int \frac{udu}{(u^2+3/4)^k} = \begin{cases} \frac{1}{2}\ln(u^2+\frac{3}{4}) + C & \text{for } k=1 \\ \frac{1}{2(1-k)}(u^2+\frac{3}{4})^{1-k} + C & \text{for } k>1. \end{cases}$$

In general, if $ax^2+bx+c$ is irreducible we can complete a square as follows

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},$$

and assuming $a>0$ we have $c - \frac{b^2}{4a} > 0$. We can then use the $u$-substitution $u = x + \frac{b}{2a}$.

**Exercise:** Compute $A, B, C, D, E$ and $F$ in (1).

**Exercise:** Compute the integral

$$\int \frac{dx}{(x-2)(x^2+2x+5)^2}.$$