Compound Interest and derivatives of exponential functions
Compound Interest (interest compounded periodically)

Start with \( P \) and an annual rate \( r \) as before.

Divide the year into \( m \) periods of equal duration and assume that the interest is compounded \( m \) times a year.

Apply the Simple Interest model to each conversion period using the rate \( \frac{r}{m} \).

Accumulated Amount after \( k \)-periods = Principal at the beginning of the \((k+1)\)-st period.

After 1-period:
\[
P(1 + \frac{r}{m})
\]

After 1-year
\[
P(1 + \frac{r}{m})^m
\]

After \( t \)-years
\[
P(1 + \frac{r}{m})^{mt}
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After 1-period: $P \left(1 + \frac{r}{m}\right)$

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Example

If \( r = 0.02 \), what results in a higher Accumulated Amount, compounding bi-annually or quarterly?

\[
P \rightarrow P \left(1 + 0.02 \right)^{mt}
\]

Graph of \( f(m) = \left(1 + 0.02 \right)^{m} \)

Compound Interest and derivatives of exponential functions
If $r = 0.02$, what results in a higher Accumulated Amount, compounding bi-annually or quarterly?
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Example

$f(m)$ is increasing $f(4) > f(2)$

Quarterly is better than bi-annually

In general one obtains a marginally better return if one compounds interest more frequently.
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Compound Interest and derivatives of exponential functions
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- In general one obtains a marginally better return if one compounds interest more frequently.
Present Value and Effective Interest

Present Value of an investment

\[ P = A \left(1 + \frac{r}{m}\right)^{-mt} \]

Effective Interest

\[ r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1 \]

\[ A = P \left(1 + r_{\text{eff}}\right)^t \]

Compound Interest and derivatives of exponential functions
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Example

If interest is compounded quarterly at a rate of 5%, what is the principal needed in order for the Accumulated Amount to be $3000 over a period of 5 years?

What about 10 years?

5 years: 2340.26

10 years: 1825.24.

Compound Interest and derivatives of exponential functions
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Interest compounded continuously

\[ A = P \left(1 + \frac{r}{m}\right)^{mt} \]

Let \( m \to \infty \).

\[ \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} = e^{rt} \]

Model of interest compounded continuously

Compound Interest and derivatives of exponential functions
Interest compounded continuously

- Compound “more and more often”.

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Example

If interest is compounded continuously at a rate of 5%, what is the principal needed in order for the Accumulated Amount to be $3000 over a period of 5 years? What about 10 years?

▶ 5 years: 2336.402
▶ 10 years: 1819.592.

Compare to the quarterly case

▶ 5 years: 2340.26
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Compound Interest and derivatives of exponential functions
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\hline
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Derivative of Exponential Functions

\[ \frac{d}{dx} e^x = \lim_{h \to 0} \left( e^x + h - e^x \right) h \]

\[ \frac{d}{dx} e^x = e^x \lim_{h \to 0} \left( e^h - 1 \right) h \]

▶ What is \( \lim_{h \to 0} \left( e^h - 1 \right) h \)?

Compound Interest and derivatives of exponential functions
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\[ \frac{d}{dx} e^x = \lim_{h \to 0} \left( \frac{e^{x+h} - e^x}{h} \right) \]
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Compound Interest and derivatives of exponential functions
Using $R$

Compound Interest and derivatives of exponential functions
Using R

```r
> for (i in 1:100){
+ z=i*(exp(1/i)-1);
+ print(paste("Iteration",i,"","","",z))}
> |
```
Using \( R \)

Compound Interest and derivatives of exponential functions
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Graph of \( x(e^{1/x} - 1) \)
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What is \( \lim_{h \to 0} \left( e^h - 1 \right) \)?

Ans: The limit equals 1.

\[ \Rightarrow \frac{d}{dx} e^x = e^x. \]
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➢ What is \( \lim_{h \to 0} \left( \frac{e^h - 1}{h} \right) \)?

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