Reduction and Equivalence of Nonlinear Distributed Symmetric Control Systems

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***Abstract*— This paper considers the 'reduction' problem for large-scale distributed control systems. In particular, we consider control-theoretic concepts for control systems containing multiple instances of identical controllers or components where the overall system is invariant with respect to interchanging these identical components. The main results are invariance of controllability, motion planning and optimal control properties for an equivalence class of symmetric systems of this type.**

I. INTRODUCTION

This paper considers nonlinear control-theoretic properties of large-scale distributed systems, which consist of, perhaps many, interconnected subsystems. Since the size of these systems can make analysis difficult or intractable, the aim of this work is to exploit symmetry properties of such systems to reduce their complexity. Unlike most model reduction problems, the approach here is *exact* in that some control-theoretic properties are equivalent in the reduced order model and large model.

The type of symmetry we consider is when certain subsystems of the overall system can be interchanged with other subsystems without changing the dynamics of the overall system. The general idea is that a distributed system is comprised of sets of multiple, repeated instances of identical hardware, which naturally can be interchanged. We represent such symmetric distributed systems using a graph-theoretic representation, as illustrated in Figure 1. Each node of the graph represents a subsystem of the overall system, and if each of the nodes 2-11 are identical, the system is characterized by an S_{10} symmetry (the symmetric group of order 10), which is a consequence of the fact that each of the subsystems 2-11 can be interchanged without altering the system.

When a subsystem is interchanged, the input/output connections of the subsystem must match the input/output connection of the replaced subsystem. For example, consider a team of mobile robots working together to manipulate an object. Suppose that each robot transmits its horizontal position to one neighboring robot and its vertical position to another. Clearly, if this robot is to be replaced by a similar robot, the system would only work if the correct, *i.e.*, horizontal or vertical, position is transmitted to the correct neighbor since each of the neighbors are expecting and acting on a particular type of information. For the system in Figure 1, for example, node 2 must interact with node 3 in the same manner that node 6 interacts with node 7, otherwise the dynamics of the overall system will be altered if they were interchanged.

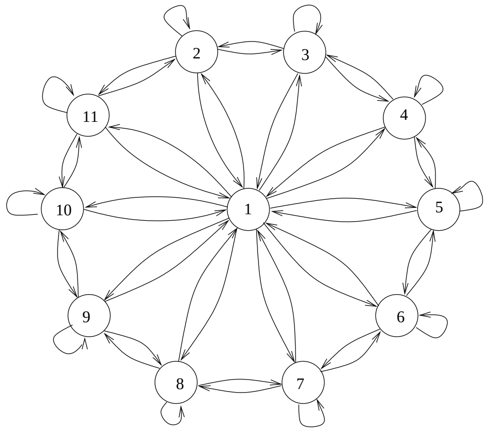

Fig. 1. An eleven node distributed system.

The method presented in this paper constructs a formal means of determining whether subsystems can be interchanged without altering the global system characteristics. Furthermore, this method can be used to then determine if a symmetric subsystem can be added without altering some of the control-theoretic properties of the system. In such a case, computations involving the "small" symmetric systems will provide a means to determine these properties for larger, symmetric systems.

There have been many efforts toward controllability of distributed systems [3], [5], [14], [1] and distributed systems with symmetry [6], [15]. These efforts are limited to *linear* systems; however, this paper considers fully nonlinear systems. There have also been many efforts toward reducing nonlinear mechanical and control systems [8], [9], [7], [10], [11], [2], [20], [21]. A similar approach was considered by Tanaka [19], [18], [17]; again, those results are limited to *linear* controllability, as opposed to the full nonlinear controllability considered in this paper.

II. DRIFTLESS SYMMETRIC NONLINEAR DISTRIBUTED SYSTEMS

A. Nonlinear Distributed Systems

This overview is based upon our previous results in [12]. We will consider smooth analytic driftless systems of the form

$$\begin{aligned} \Sigma: \quad \dot{x} &= g_{1,1}(x)u_{1,1} + g_{1,2}(x)u_{1,2} + \cdots \\ &+ g_{2,1}(x)u_{2,1} + g_{2,2}(x)u_{2,2} + \cdots \\ &\vdots \end{aligned} \qquad (1)$$

$$+ \quad g_{n,1}(x)u_{n,1} + g_{n,2}(x)u_{n,2} + \cdots \quad x \in M,$$

where M is a smooth manifold and $g_{i,j}$ are smooth analytic vector fields on M. , and $u = \{u_1, \ldots, u_n\} \in \mathcal{U}$, where $\mathcal{U}$ is the set of admissible controls. We assume that the set of admissible controls is a subset of $\mathbb{R}^n$ such that,

$$\text{Aff}(\mathcal{U}) = \mathbb{R}^n,$$

where Aff($\mathcal{U}$) denotes the affine hull of $\mathcal{U}$. Since we are considering distributed systems, the system is assumed to be organized into subsystems, corresponding to which are certain vector fields and control inputs. In Equation (1), the first subscript on the g's and u's indexes the subsystem to which the vector field and control input corresponds, and the second subscript indexes different vector fields and inputs within that subsystem. To avoid notational clutter, if a vector field only has one subscript, *i.e.*, $g_i(x)$, then it represents the *ordered set* of vector fields associated with node i, *i.e.*, $g_i(x) = \{g_{i,1}, g_{i,2}, \ldots\}$. Similarly, u_i would represent the ordered set $u_i = \{u_{i,1}, u_{i,2}, \ldots\}$. Any property defined for single-subscripted vector field is understood to apply to each member in that set.

Elaborating further on the distributed nature of the system, we assume that M is partitioned into a set of m regular submanifolds, M_i such that M is the Cartesian product of the M_i, *i.e.* $M = \prod_{i=1}^{m} M_i$. Each submanifold M_i represents a *subsystem, module, node* or *component* of the distributed system (all these terms will be used interchangeably). For example, in a system of cooperating robots, each M_i would represent the configuration space for one robot in the system and $\{u_{i,1}, u_{i,2}, \ldots\}$ would be the control inputs for that robot.

Since, it is often the case that the dynamics of any one module or node is only affected by its own controller and states as well as the control inputs and states of a limited subset of the other nodes (usually its neighbors) and to help aid in providing a clear presentation, we will utilize a graph-theoretic representation of distributed systems. Formally, we define the digraph of a nonlinear control system Σ, written as $\mathcal{G}_\Sigma$, to be the pair (**V**,**E**) consisting of a set of vertices $\mathbf{V} = \{V_1, \ldots, V_m\}$ and the set of edges, denoted by **E**, which are ordered pairs of elements of **V**. Each *vertex* represents one module M_i, *i.e.*, $V_i = M_i$. The *edge directed from* V_i *to* V_j, $E_{i,j} = \{V_i, V_j\} \in \mathbf{E}$, represents a vector field which maps elements of the vertices V_i and V_j to the tangent space of the end-point vertex V_j *i.e.*,

$$E_{i,j} : V_i \times V_j \to TV_j.$$

The edge $E_{i,j}$ is the sum of the jth components of the $g_{i,k}(x)$'s from Equation 1 that multiply the control inputs associated with node i. If it is necessary to further distinguish the edges by representing to which vector field within the subsystem it is associated, a third subscript can be added, *i.e.*,

$$E_{i,j,k} : V_i \times V_j \to TV_j.$$

This edge, $E_{i,j,k}$, still maps between the same spaces, but the third subscript indicates that it is the jth component of $g_{i,k}$. Again, to avoid unnecessary notational complexities, we will often drop the third subscript (indexing to which control input in node i the vector fields is associated) and use $E_{i.j}$ to represent the *ordered set* of vector fields, $E_{i,j} = \{E_{i,j,1}.E_{i,j,2}, \ldots\}$.

Let $\tilde{V}_i = \{V_{\tilde{i}_1}, \ldots, V_{\tilde{i}_m}\}$ be an ordered set of vertices which are connected to V_i by edges directed from V_i to the elements of $\tilde{V}_i$ and let $\tilde{E}_i = \{E_{i,\tilde{i}_1}, \ldots, E_{i,\tilde{i}_m}\}$ be an ordered set of edges directed from V_i to elements $\tilde{V}_i$. The manner by which $\tilde{V}_i$ and $\tilde{E}_i$ are ordered is determined by interactions and/or communications between nodes. Note that ordering $\tilde{V}_i$ imposes some topological structure on the system; in particular, for nodes that can be interchanged, the $\tilde{V}_i$ sets must be ordered identically with respect to their neighbors so that their interactions with adjacent nodes are the same before and after they are interchanged to maintain invariance of the overall system dynamics.

B. *Symmetric nonlinear distributed systems*

Now we will consider what it means for a nonlinear distributed system to be symmetric. This will be represented by the fact that vector fields from various nodes will, in some sense, be equivalent. Since the vector fields directed from different nodes are defined on different spaces, we need a definition of equivalence which is more than just requiring that they be "identical."

Definition 1: Two vector fields, g_1 And g_2 are *equivalent,* denoted $g_1 \sim g_2$, if there exists a diffeomorphism, $\psi : M \mapsto M$, such that

$$\psi_* \circ g_1(W) = g_2(\psi(W))$$

where W is an open set. Equivalently, we can define $E_{i,j} \sim E_{k,l}$ by only considering the jth and lth components of g_i and g_k, respectively.

The definition of vector field equivalence applies to general submanifolds without any assumptions regarding the relationship between the coordinate systems defined on different nodes; however, often each node will be designed with a complimentary coordinate system so that the diffeomorphism, ψ, in definition 1 is a simple permutation of states and the open set, W, is the whole domain of validity of the system equations. Equivalence among vector fields can often be determined by inspection; however, this inspection is typically on an edge–by–edge basis in contrast to the computational approach involving the full control vector field.

Recall that the symmetric group of order $p!$, denoted S_p, is the group of permutations of p objects and that such a permutation of a set $X = \{1, \ldots, p\}$ is a one-to-one mapping of X onto itself. Such a permutation ρ is written

$$\rho = \begin{pmatrix} 1 & 2 & \cdots & p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$$

which represents that 1 is mapped to k_1, 2 is mapped to k_2, *etc*. Given an equivalence relation among vector fields, we now define a symmetric nonlinear distributed system.

Definition 2: Let a *symmetry orbit*, $\mathbf{O} \subset \mathbf{V}$, be a subset of $\mathbf{V}$ containing p vertices, *i.e.,* $\mathbf{O} = \{V_{k_1}, V_{k_2}, \ldots, V_{k_p}\}$, let $\mathbf{F} = \mathbf{V} \setminus \mathbf{O}$ be the subset of V containing $n - p$ *fixed* vertices, *i.e.,* $\mathbf{F} = \{V_{f_1}, \ldots, V_{f_{n-p}}\}$, let $\tilde{V}_{k_l}$ be the ordered set of vertices connected to V_{k_l}, and let $\rho \in S_p$. The system Σ is a *symmetric nonlinear distributed system* if

$$g_{k_i} \sim g_{\rho(k_i)} \qquad \forall i \in \{1, \ldots, p\} \text{ and } \forall \rho \in S_p.$$

Equivalently, a system is a symmetric nonlinear distributed system if

$$E_{k,\tilde{k}_l} \sim E_{\rho(k)_l,\rho(\tilde{k})_l} \quad \text{and} \quad E_{\tilde{k}_l,k} \sim E_{\rho(\tilde{k}),\rho(k)_l},$$

$\forall k \in \{k_1, \ldots, k_p\}$, $\forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$, and $\forall \rho \in S_p$.

Before we define nonlinear symmetric system equivalence, we need to develop a technique which allows us to compare the relative size of two systems. Let Σ_1 and Σ_2 be symmetric nonlinear distributed systems and let $\mathcal{G}_{\Sigma_1} = \{\mathbf{V_1}, \mathbf{E_1}\}$ and $\mathcal{G}_{\Sigma_2} = \{\mathbf{V_2}, \mathbf{E_2}\}$ denote their corresponding digraphs. We say that $\mathcal{G}_{\Sigma_1} \geq \mathcal{G}_{\Sigma_2}$ if the number of vertices in $\mathcal{G}_{\Sigma_2}$ is greater than the number of vertices in $\mathcal{G}_{\Sigma_2}$. Now nonlinear distributed system equivalence is defined as follows.

Definition 3: Let Σ_1 and Σ_2 be symmetric nonlinear distributed systems and $\mathcal{G}_{\Sigma_1} \geq \mathcal{G}_{\Sigma_2}$. Since each system is a symmetric nonlinear distributed system there exist symmetry orbits $\mathbf{O_1} \subset \mathbf{V_1}$ and $\mathbf{O_2} \subset \mathbf{V_2}$ containing p and q ($p \geq q$) vertices, respectively, *i.e.,*

$$\mathbf{O_1} = \{V_{(k_1)_1}, V_{(k_1)_2}, \ldots, V_{(k_1)_p}\}$$

and

$$\mathbf{O_2} = \{V_{(k_2)_1}, V_{(k_2)_2}, \ldots, V_{(k_2)_q}\}.$$

The systems Σ_1 and Σ_2 are *equivalent symmetric nonlinear distributed systems* if

1) $E_{k,(\tilde{k}_l)_1} \sim E_{k,(\tilde{k}_l)_2} \quad \forall k \in \{k_1, \ldots, k_q\}, \forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$
2) $\mathbf{F_1} = \mathbf{V_1} \setminus \mathbf{O_1}$ and $\mathbf{F_2} = \mathbf{V_2} \setminus \mathbf{O_2}$ contain the same number of vertices, *i.e.,* $\mathbf{F_1} = \mathbf{F_2} = \{V_1, \ldots, V_m\}$, and
3) $E_{k,(\tilde{k}_l)_1} \sim E_{k,(\tilde{k}_l)_2} \quad \forall k \in \{1, \ldots, m\}, \forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$.

Denote the equivalence class of systems defined by this equivalence relation by $\bar{\Sigma}$.

Equivalence between symmetric nonlinear distributed systems requires that every member have an equivalent input/output structure and the same number of fixed nodes. Furthermore, corresponding elements of $\tilde{E}_i$ in each system must be vector field equivalent. Note, not all digraphs have the same number of vertices and edges, so the comparison is only between elements that exist in each digraph. To illustrate the notation used in the definition of system equivalence, consider the following example.

III. Results

This section presents three main classes of results for symmetric distributed systems. The first is related to controllability, the second to a constructive motion planning algorithm and the third to preliminary results for optimal control of such systems.

A. Nonlinear Controllability

Given an open set $W \subseteq M$, define $R^W(x_0, T)$ to be the set of states x such that there exists $u : [0, T] \to \mathcal{U}$ that steers the control system from $x(0) = x_0$ to $x(T) = x_f$ and satisfies $x(t) \in W$ for $0 \leq t \leq T$, where $\mathcal{U}$ is the set of admissible controls. Define

$$R^W(x_0, \leq T) = \bigcup_{0<\tau\leq T} R^W(x_0, \tau). \qquad (2)$$

We will refer to $R^W(x_0, \leq T)$ as the set of states reachable up to time T.

Definition 4: A system is *small time locally controllable* ('STLC', or simply 'controllable') if $R^W(x_0, \leq T)$ contains a neighborhood of x_0 for all neighborhoods W of x_0 and $T > 0$.

Let $\mathcal{C}$ denote the smallest subalgebra of $V^\infty(M)$ (the Lie algebra of smooth vector fields on a manifold M whose product is the Lie bracket, $[\cdot, \cdot]$) that contains $g_1, \ldots, g_m$. If $\dim(\mathcal{C}) = \dim M$ at a point x, then the system described by Equation 1 satisfies the *Lie Algebra Rank Condition* ('LARC') at x. The following is well known as 'Chow's Theorem.'

Theorem 5: If the system described by Equation (1) satisfies the LARC at a point x_0 then it is STLC from x_0.

The following is the main controllability result.

Proposition 6: If any one member, Σ_n, of the equivalence class of symmetric distributed control systems, $\overline{\Sigma}$ is STLC, then all members of the equivalence class, $\Sigma_i \in \overline{\Sigma}$ where $i > n$ of symmetric distributed control systems are STLC.

The proof is a straight-forward construction that makes use of the fact that diffeomorphisms are natural with respect to Lie brackets. A similar theorem for nonlinear systems with drift based on the usual good/bad bracket test due to Sussmann [16] is similarly obtained.

B. Nonholonomic Motion Planning

Symmetries may be exploited in distributed systems for motion planning purposes. Space limitations prevent their inclusion here. An interested reader is referred to [13].

C. Optimal Control

The ultimate goal for considering the optimal control problem are similar reduction results, *i.e.,* solving the optimization problem for a smaller system and using the results for a larger system. Initial results related to the bifurcations of optimal solutions appear in [4]. We adopt a simplified version of the robotic unicycle as a prototypical model. The simple kinematics of this kind of robot are described by

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= u_2. \end{aligned} \qquad (3)$$

The problem is to find the controls $u_{i_1}(t), u_{i_2}(t)$ for each robot i which steer a formation of robots of this type from the start configuration to its goal configuration, while maintaining a rigid body formation at the beginning and end of the trajectory and minimizing the global performance index

$$J = \int_0^{t_f} \sum_{i=1}^{n} \left((u_{i_1})^2 + (u_{i_2})^2 \right) + \sum_{i=1}^{n-1} k \left(d_i - \overline{d} \right)^2 dt$$

subject to the robotic kinematic constraints in Equation 3, where $n > 2$ is the number of robots, $d_i = ((x_i - x_{i+1})^2 + (y_i - y_{i+1})^2)^{1/2}$ is the Euclidean distance from ith to $(i+1)$th robots, $\overline{d}$ is the desired distance between two adjacent robots, and k is a non-negative weighting constant. The cost function minimizes a combination of the control effort (first summation) and the deviation from a desired formation (second summation). Bifurcations in the nature and the form of the solutions are illustrated in Figure 2 for a system of seven robots.

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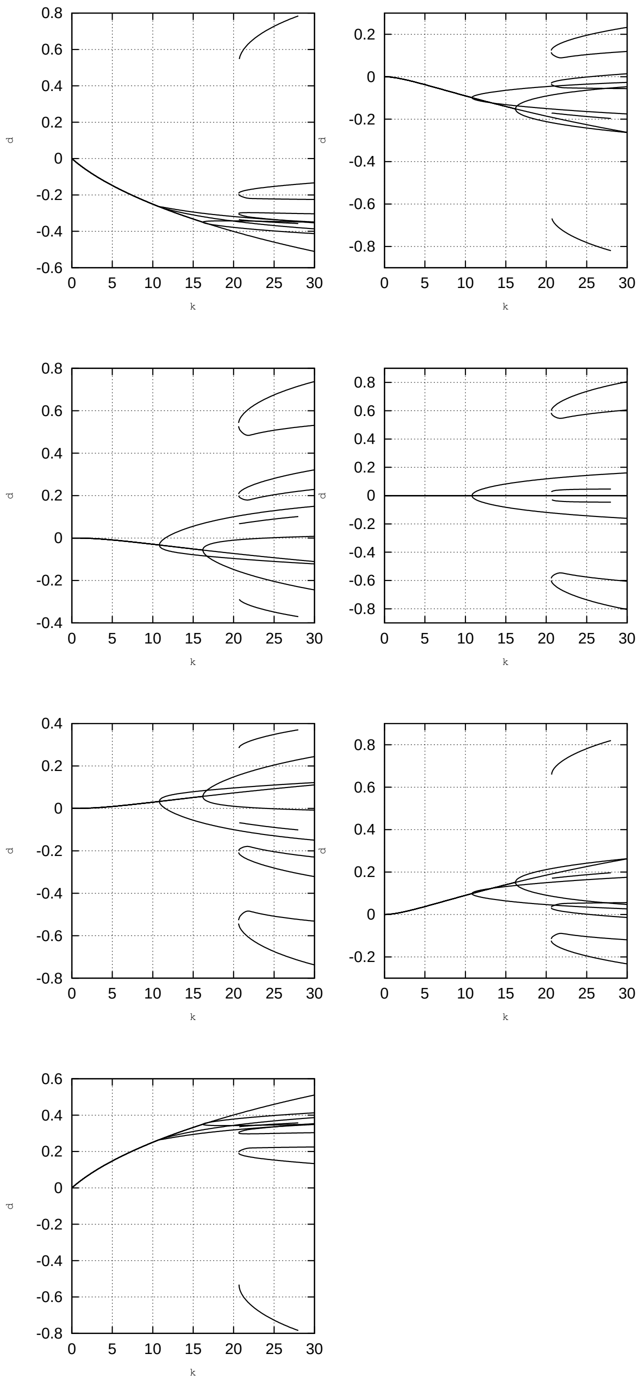

Fig. 2. Bifurcation diagrams for a 7-robotic system, robots 1 through 7, respectively.

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