Finite-Time Lyapunov Analysis and Optimal Control

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***Abstract*— Hyper-sensitive optimal control problems present difficulty for general purpose solvers. A numerical implementation of an approach analogous to the method of matched asymptotic expansions requires determining initial and final conditions on appropriate invariant manifolds to sufficient accuracy. Finite-time Lyapunov exponents and vectors are employed for this purpose. The approach is explained and illustrated in the context of a simple transparent example.**

I. Introduction

Numerical methods for solving optimal control problems (OCPs) can be divided into two main categories, direct and indirect. Indirect methods involve solving the associated Hamiltonian boundary value problem (HBVP) for an extremal solution that satisfies the first-order necessary conditions. A survey of direct and indirect methods, noting their advantages and disadvantages, is given in [5].

An OCP is called hyper-sensitive if the final time is large relative to some of the contraction and expansion rates of the associated Hamiltonian system [13], [14]. The solution to a hyper-sensitive problem can be qualitatively described in three segment as "take-off", "cruise" and "landing" analogous to optimal flight of an aircraft between distant locations. The "cruise" segment is primarily determined by the cost function and the state dynamics, whereas the "take-off" and "landing" segments are determined by the boundary conditions and the goal of connecting these to the "cruise" segment. As the final time increases so does the duration of the cruise segment which shadows a slow reduced-order manifold. When the final time is large, the sensitivity of the final state to the unknown initial conditions makes the HBVP ill-conditioned. The ill-conditioning can be removed by approximating the solution with a composite one: a concatenation of boundary-layer solutions (take-off/landing segments) with a solution segment on the slow manifold (cruise segment). The completely hyper-sensitive case is a degenerate case for which the 'cruise segment' is near-equilibrium motion, and rather than a slow manifold, there is an equilibrium point. The more general case in which the cruise segment shadows a trajectory on the slow manifold is called partially hyper-sensitive.

Solution approximation for completely hyper-sensitive optimal control problems, based on the geometric structure of the associated Hamiltonian dynamics, has been addressed in [2], [13]. The solution to the HBVP is such that the solution in the initial boundary layer is approximated by a trajectory on the stable manifold of the equilibrium point, the solution in the final boundary layer is approximated by a trajectory on the unstable manifold of the equilibrium, and the boundary layer solutions are approximately matched at the equilibrium point. The focus of the present paper is on determining the unknown boundary conditions such that the solution end points lie on the appropriate invariant manifolds to sufficient accuracy.

Rather than use information related to the equilibrium point which would not be available in the partially hyper-sensitive case, our approach uses a dichotomic basis for the phase space tangent bundle to define conditions satisfied by points on the stable and unstable manifolds of an equilibrium point. Dichotomy transformations have been used to solve boundary-value problems for ordinary differential equations [3], and to solve fixed end-point optimal control problems [2], [17]. Chow used dichotomy transformations to solve nonlinear HBVPs with linear boundary-layer dynamics [7]. For nonlinear HBVPs, a dichotomic basis has been approximated using eigenvalues and eigenvectors [13] and finite-time Lyapunov exponents and vectors [16]. The latter information can provide greater accuracy and is more generally applicable [11], [12]. Finite-time Lyapunov exponents, and in some cases vectors, have also been used to analyze fluids [8], [9], [15] and atmospheric circulation [6].

II. Finite-Time Lyapunov Analysis

In previous work [11] and [12] finite-time Lyapunov analysis (FTLA) was applied to autononmous nonlinear dynamical systems to define and diagnose two-timescale behavior and compute points on a slow manifold, if one exists. The approach is to decompose the tangent bundle into subbundles on the basis of the characteristic exponential rates for the associated linear flow, and then to translate the tangent bundle structure into manifold structure in the base space. In FTLA, the characteristic exponential rates and associated directions are given, respectively, by finite-time Lyapunov exponents (FTLEs) and finite-time Lyapunov vectors (FTLVs). This approach has been guided by the asymptotic theory of partially hyperbolic invariant sets [4]. The finite-time tangent bundle decomposition can be viewed as an approximation of the asymptotic Oseledec's decomposition [4]. It has been established in [12] that under certain conditions the finite-time decomposition approaches the (suitably defined) asymptotic decomposition exponentially fast, the

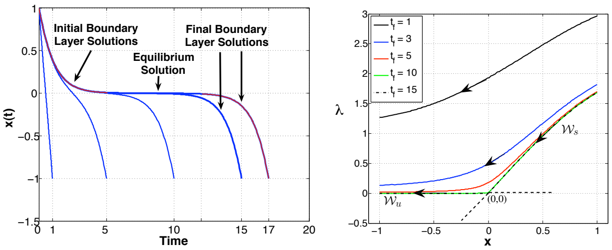

Fig. 1. Optimal solutions from GESOP© for different final times, $t_f = 1, 5, 10, 15$ and 17, where $x(0) = 1$ and $x(t_f) = -1$ in the time domain (left) and in the phase space (right).

rate being given by the size of the gaps in the spectrum of the FTLEs.

III. Solution Approximation Strategy for a Completely Hyper-Sensitive Problem

We use a simple transparent example to present the approximate solution strategy, and later to address some implementation issues, for a completely hyper-sensitive optimal control problem. Consider Lam's optimal control problem [10]: determine the control u^* and corresponding state trajectory x^* that minimize the cost

$$\mathrm{J} = \frac{1}{2}\int_0^{t_f} u^2 dt \tag{1}$$

subject to the dynamic constraint

$$\dot{x} = \sin x + u \tag{2}$$

for a given final time t_f and boundary conditions $x(0) = 1$ and $x(t_f) = -1$. The first-order necessary conditions lead to the following Hamiltonian boundary value problem (HBVP)

$$\begin{aligned} \dot{x} &= \sin x - \lambda \\ \dot{\lambda} &= -\lambda \cos x \\ x(0) = 1, \quad & x(t_f) = -1 \end{aligned} \tag{3}$$

for extremal solutions. Defining the Hamiltonian $H = (1/2)u^2 + \lambda(sinx + u)$ and the phase $p = (x, \lambda)^T$, we can write the state/costate dynamics in (3) in the form $\dot{p} = h(p)$, where $h = (\partial H/\partial\lambda, -\partial H/\partial x)^T$, to show that we are dealing with a Hamiltonian system.

We have solved the OCP using the optimization program GESOP© [1] for different final times; see Fig. 1. GESOP has several options, of these we used the direct multiple shooting method. The optimal trajectory and control are determined. Using the necessary condition, $u^* = -\lambda$, the solutions can be plotted in the (x, λ) phase plane. We see that as the final time gets larger the solution trajectories shadow more and more closely branches of the stable, $\mathcal{W}^s$, and unstable, $\mathcal{W}^u$, invariant manifolds of the equilibrium point, p_{eq}, at $(0, 0)$, and the solution spends more and more time near the equilibrium point. As t_f gets beyond 17, obtaining a numerical solution with GESOP© gets more and more difficult. In contrast, as t_f increases, the following approximate solution becomes more and more accurate and no harder to obtain.

For sufficiently large final times, the optimal strategy can be viewed in phase space as getting on $\mathcal{W}^s(p_{eq})$ where $x(0) = 1$ and steering along it to the equilibrium point and then steering along $\mathcal{W}^u(p_{eq})$ to reach the point on it where $x(t_f) = -1$. Consistent with this viewpoint, the solution to a completely hyper-sensitive OCP, $p^*(t)$, can be approximated by the composite function

$$\hat{p}(t) = \begin{cases} \hat{p}_s(t) & 0 \le t \le t_{ibl} \\ p_{eq} & t_{ibl} < t \le t_{fbl} \\ \hat{p}_u(t) & t_{fbl} \le t \le t_f \end{cases} \tag{4}$$

where $\hat{p}_s(t)$ is the approximate initial boundary-layer solution for $t \in [0,\ t_{ibl}]$ with the initial condition on the stable manifold, i.e., $\hat{p}_s(0) = (x_0, \lambda_0) \in \mathcal{W}^s(p_{eq})$; p_{eq} is the equilibrium solution which approximates the slow "cruise" segment; and $\hat{p}_u(t)$ is the approximate final boundary-layer solution for $t \in [t_{fbl},\ t_f]$ with the final condition on the unstable manifold, i.e., $\hat{p}_u(t_f) = (x_{t_f}, \lambda_{t_f}) \in \mathcal{W}^u(p_{eq})$. The solutions in the boundary-layers can be constructed by integrating, in forward and backward time respectively, the Hamiltonian dynamics from initial and final phase points on the corresponding invariant manifolds that satisfy the boundary conditions. The composite approximate solution is obtained by concatenating the boundary-layer solutions with the equilibrium solution. The times, t_{ibl} and t_{fbl}, defining the initial and final boundary-layer durations are selected such that $\hat{p}_s$ and $\hat{p}_u$ reach the equilibrium point up to a specified accuracy in forward and backward time respectively.

The primary challenge in developing this approach is to determine the unknown boundary conditions such that the initial and final phase points are sufficiently close to $\mathcal{W}^s(p_{eq})$ and $\mathcal{W}^u(p_{eq})$ respectively. The choice to base our approach on finite-time Lyapunov exponents and vectors (FTLE/Vs), rather than use other methods particularly suited to the structure near the equilibrium point, is driven by the goal of extending the approach to the partially hyper-sensitive case.

IV. Diagnosing Hyper-Sensitivity

If numerical solution of an OCP using a software package such as GESOP© proves difficult and reducing the final time alleviates the difficulty, hyper-sensitivity should be investigated. By observing how the solution evolves as t_f is varied, the relevant phase space region can be identified. Computing the FTLE spectrum at selected phase points in this region can quantify the exponential rates. If the spectrum uniformly separates into fast stable, slow, and fast unstable subsets, and the 'fast' rates are indeed fast relative to the time interval of interest, then hyper-sensitivity is confirmed. To describe the general case, let n be the dimension of the state dynamics; then it follows that $2n$ is the dimension of the associated Hamiltonian system. The spectrum also reveals the equal dimensions, n^{fs} and n^{fu}, of the fast stable and fast unstable behavior, respectively. If $n^{fs} + n^{fu} = 2n$, then the OCP is completely hyper-sensitive for sufficiently large t_f. If $n^{fs} + n^{fu} < 2n$, then the OCP is partially hyper-sensitive for sufficiently large t_f.

An important issue is how to select T, the averaging time. Lyapunov exponents are averages of the pointwise exponential rates, i.e., the averages of the appropriate kinematic eigenvalues (KEs) [3], over a segment of a trajectory of the Hamiltonian system. Points of the trajectory segment can be indexed by elapsed time. As long as the trajectory evolves in a phase space region where the KEs are uniform, larger averaging time T allows further progress in convergence of both the FTLEs and the FTLVs and better information. However once the trajectory enters a region of different KEs this ceases to be true. In Lam's problem, there is a qualitative difference between the local FTLEs and the asymptotic Lyapunov exponents. The two asymptotic Lyapunov exponents are zero at all points except points on the heteroclinic orbits connecting the equilibria. On the other hand, the FTLEs can indicate the hyperbolic nature of the dynamics in a neighborhood of a stable or unstable manifold associated with an equilibrium point. This is also the observation that led to the maximum FTLE method that has been used to identifying Lagrangian coherent structures in fluids [8], [9], [15].

A systematic approach to selecting the averaging time T such that the FTLEs indicate the local nature of the flow to sufficient accuracy, for a flow with non-uniform kinematic eigenvalues is as follows. Fig. 2 shows trajectories that shadow the invariant manifolds of the equilibria. The KE for the vector field $h(p)$, given by

$$\rho_h(t) = \frac{1}{2} h^T(p)((Dh(p))^T + Dh(p))h(p) \qquad (5)$$

was computed along these trajectories. The red circles mark points where the ρ_h is zero; it is positive on one side of such a point and negative on the other. The positive sign indicates that neighboring points on the trajectory are separating with time, and negative sign indicates that they are getting closer to each other. Thus the red circles mark the boundaries separating regions of uniform (in sign) ρ_h. The time T should be selected to average as long as possible over a uniform region. Fig. 3 shows for a grid of $\lambda(0)$ values, the trajectories that evolve in forward time from the corresponding initial phase points. The zero-crossing for ρ_h are indicated, as are the times T over which the FTLE/Vs can be computed to determine the stable subspace. Although some of the trajectories begin in a different uniform region, most of the time stated is spent in the uniform region where the attraction to the equilibrium point is sensed. Fig. 4 shows that with proper selection of T, the maximum FTLE contours identify the stable and unstable manifolds of $(0, 0)$.

V. Approximate Solution Construction

To implement the solution approximation strategy, we need to compute $\lambda(0)$ such that the point $p(0) = (x(0), \lambda(0))^T$ is on the stable manifold $\mathcal{W}^s$. The FTLA approach is to determine the point $p(0)$ such that $h(p(0)) \in E^s(p(0))$, where $E^s(p(0))$ is the stable subspace approximation by the span of appropriate Lyapunov vectors. To accomplish this, we determine $\lambda(0)$ such that $\langle h(p(0)), w\rangle = 0$ for all $w \in E^s(p(0))^{\perp}$.

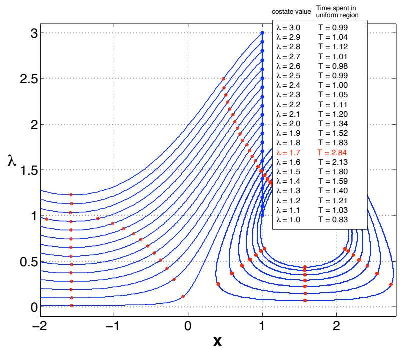

Fig. 3. Trajectories starting at $p(0) = (1, \lambda(0))$ for a grid of $\lambda(0)$ values. On each trajectory the zero ρ_h points are noted. The maximum averaging time is noted for each value of $\lambda(0)$.

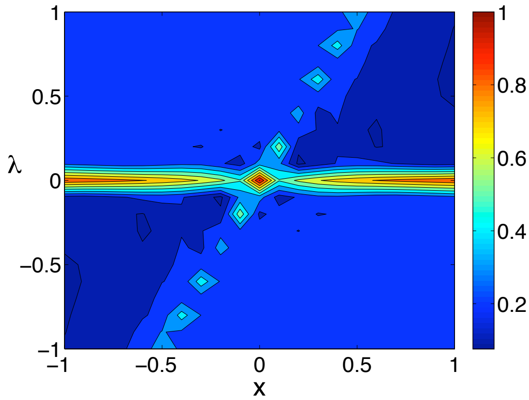

Fig. 4. Level contours of the FTLE field for the Hamiltonian system in (3) on a grid of points $\mathcal{M} = \{(x, \lambda) \in \mathbb{R}^2 | \; x \in [-1,\; 1],\; \lambda \in [-1,\; 1]\}$.

Fig. 2 shows the 'lines of ambiguity' at $x = -\pi/2$ and $x = \pi/2$. At these values of x, the orthogonality condition is satisfied for all values of λ and cannot be used to identify the particular value of λ that would place p on the appropriate invariant manifold. For all other values of x in this range, there are isolated solutions to the orthogonality condition corresponding to the desired manifolds.

An analogous procedure is followed to place the final phase point on the unstable manifold.

Applying this approach, the solution approximation to Lam's optimal control shown in Fig. 5 was obtained.

VI. Partially Hyper-Sensitive Optimal Control Problems

The consideration of completely hyper-sensitive OCPs in this paper is just a stepping stone to the consideration of partially hyper-sensitive OCPs. Partially hypersensitive optimal control problems are associated with HBVPs that have fast and slow behavior. Ill-conditioning is only associated with certain directions. An accurate approximation to the solution of a partially hyper-sensitive problem can be constructed from three components: a short duration initial boundary-layer segment, a long duration slow segment, and a short duration terminal

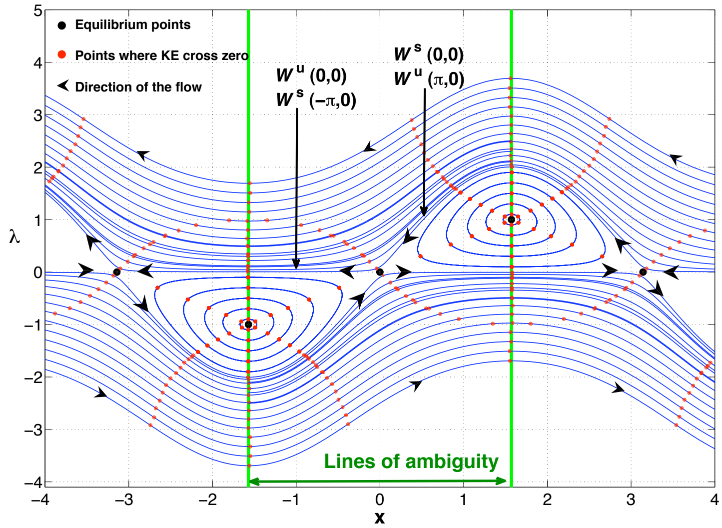

Fig. 2. Hamiltonian phase space of the system given in (3) showing stable and unstable manifolds of the equilibria, trajectories shadowing these manifolds, regions of uniformity (boundaries denoted by green circles), and lines of ambiguity.

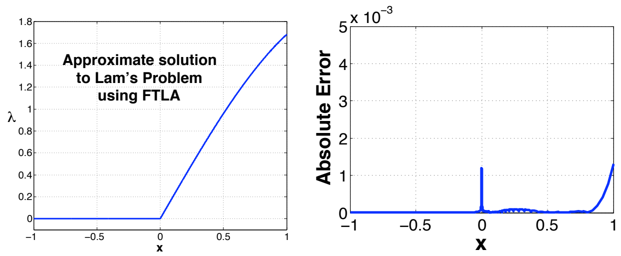

Fig. 5. **Left:** Approximation to Lam's problem using FTLA. The approximation to the stable invariant manifold is constructed with re-initializations for every T=0.1. **Right:** The absolute error between the optimal solution from GESOP and the approximate solution constructed by FTLA vs x.

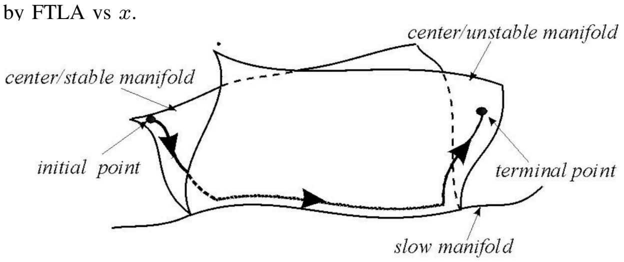

Fig. 6. Geometry of solution to partially hyper-sensitive optimal control problem in Hamiltonian phase space. Slow manifold should be 2D.

boundary-layer segment; loosely analogous to the method of matched asymptotic expansions. In this manner, the hyper-sensitivity can be avoided. Constructing segments requires locating points on invariant manifolds. Rather than stable and unstable manifolds of an equilibrium, in the partially hyper-sensitive case the manifolds of interest are center-stable and center-unstable. Fig. 6 illustrates the manifold structure, showing the initial and final boundary-layers and the intermediate segment that shadows a slow invariant manifold. Because we are dealing with a Hamiltonian system the phase space would be 4-dimensional and the slow manifold would be 2-dimensional, though the figure does not accurately depict the dimension of the slow manifold.

REFERENCES

[1] *www.gesop.de*.

[2] B.O. Anderson and P.V. Kokotovic, *Optimal control problems over large time intervals*, Automatica **23** (1987), no. 3, 355 – 363.

[3] U.M. Ascher, R.M. Mattheij, and R.D. Russell, *Numerical solution of boundary value problems for ordinary differential equtions*, SIAM press, Philadelphia, 1995.

[4] L. Barreira and Y.B. Pesin, *Lyapunov exponents and smooth ergodic theory*, American Mathematical Society, Providence, USA, 2002.

[5] J.T. Betts, *Survey of numerical methods for trajectory optimization*, Journal of Guidance, Control, and Dynamics **21** (1998), no. 2, 193 – 207.

[6] R. Buizza and T.N. Palmer, *The singular-vector structure of the atmospheric global circulation*, J. Atmos. Sci. **52** (1995), no. 9, 1434 – 1459.

[7] J.H. Chow, *A class of singularly perturbed nonlinear, fixed end-point control problems*, J. Opt. Theory Appl. **29** (1979), 231 – 251.

[8] R. Doerner, B. Hubinger, W. Martienssen, S. Grossmann, and S. Thomae, *Stable manifolds and predictability of dynamical systems*, Chaos, Solitons and Fractals **10** (1999), no. 11, 1759 – 1782.

[9] G. Haller, *Distinguished material surfaces and coherent structures in three-dimensional fluid flows*, Physica D **149** (2001), 248 – 277.

[10] S.-H. Lam, *Personal communication*.

[11] K.D. Mease, S. Bharadwaj, and S. Iravanchy, *Timescale analysis for nonlinear dynamical systems*, J. Guidance Control and Dynamic **26** (2003), no. 1, 318 – 330.

[12] K.D. Mease, U. Topcu, and E. Aykutluğ, *Characterizing two-timescale nonlinear dynamics using finite-time Lyapunov exponents and vectors*, arXiv:0807.0239 [math.DS] (2008).

[13] A.V. Rao and K.D. Mease, *Dichotomic basis approach to solving hyper-sensitive optimal control problems*, Automatica **35** (1999), 3633–3642.

[14] ———, *Eigenvector approximate dichotomy basis method for solving hyper-sensitive optimal control problems*, Optimal Control: Applications and Methods **21** (2000), 1–19.

[15] S. C. Shadden, F. Lekien, and J. E. Marsden, *Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows*, Physica D **212** (2005), 271 – 304.

[16] U. Topcu and K.D Mease, *Using Lyapunov vectors and dichotomy to solve hyper-sensitive optimal control problems*, Proceedings of the 45^{th} IEEE Conference on Decision and Control, 2006.

[17] R.R. Wilde and P.V. Kokotovic, *A dichotomy in linear control theory*, IEEE Transactions on Automatic control **17** (1972), 382 – 383.