READING LIST FOR BERGDALL'S LECTURES AT NOTRE DAME

July 15, 2022

This is a reading list, partially annotated, for my four lectures at Notre Dame in July 2022. They follow my lectures closely and reflect my taste. Omissions are not insults.

If you start to make progress on infinite ferns of GL(n) over the rationals: let me know!

Best,
John

Lecture 1: Examples of Galois representations

In Lecture 1, I introduced the absolute Galois group of $\mathbf{Q}$ and gave many examples.

Galois theory for infinite Galois extensions can be learned from Chapter IV, Section 1 in Neukirch's text on algebraic number theory:

- J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

I do not have (in my mind) direct references for characters of Galois groups, unfortunately. (I learned them in course work). The Tate module of an elliptic curve can be read from a graduate textbook. See Section 7 of Chapter 3 in Silverman's text, (along with Chapters 5, 7-8):

- J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

Mazur's deformation space is introduced in a paper of Mazur in the late 1980's. He also wrote an article for the Fermat's Last Theorem conference in Boston. Gouvêa also wrote notes for a summer school at the Park City Mathematics Institute. In order, the references are:

- B. Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.
- B. Mazur. An introduction to the deformation theory of Galois representations. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.
- F. Q. Gouvêa. Deformations of Galois representations. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 233–406. Amer. Math. Soc., Providence, RI, 2001. Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton.

Attaching Galois representations to automorphic forms is a giant subject. The most modern papers on the subject are the paper of Lan, Harris, Taylor, and Thorne, and the paper of Scholze.

- M. Harris, K.-W. Lan, R. Taylor, and J. Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3:Paper No. 37, 308, 2016.
- P. Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.

I give the most modern references because it is difficult to "start at the beginning". I do not know accessibly written references for associating Galois representations to cuspforms on GL(2). However, for many it suffices to just understand the statement. In that case, try looking at Deligne and Serre's paper on weight one eigenforms, Serre's paper on Galois representations modulo p, and the final chapter (Chapter 9) of Diamond and Shurman's graduate text on modular forms.

- P. Deligne and J.-P. Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.
- J.-P. Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.
- F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

Lecture 2: Introduction to eigenvarieties

There is a very modern and accessible introduction to eigenvarieties, focused on $\mathrm{GL}(2)_{/\mathbf{Q}}$, written by Joël Bellaïche.

- J. Bellaïche. *The eigenbook—eigenvarieties, families of Galois representations, p-adic L-functions*. Pathways in Mathematics. Birkhäuser/Springer, Cham, [2021] ©2021.

(This text incidentally also covers the algebraicity of special values and the theory of p-adic L-functions for GL(2).)

The argument at the start of my lecture is the style of arugment you can find in Hida's work. For instance, it can be compared well with the argument in pages 203–205 of Hida's "elementary" text:

- H. Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

Calculating cohomology of locally symmetric spaces seems daunting at first. You don't see those examples in algebraic topology courses. I learned how to make explicit calculations from Hansen's paper on eigenvarieties

- D. Hansen. Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality. *J. Reine Angew. Math.*, 730:1–64, 2017. With an appendix by James Newton.

(That builds on a never published preprint of Ash and Stevens, which I don't recommend.) Bellaïche's book also gives nice calculations, focusing on the spaces of the form $\Gamma\backslash\mathfrak{h}$. The presentation in Hansen is far more adelic. A GL(2)-based adelic set of calculations can also be found in a paper I wrote with Hansen:

- J. Bergdall and D. Hansen. On p-adic L-functions for Hilbert modular forms. *To appear in Memoirs of the AMS*, 2022.

These references are just the tip of the iceberg. Many other names: Dimitrov, Ghate, Raghuram, ... come to mind to have written papers with explicit cohomological calculations in them. Not the mention Hida one more time.

Other than Bellaïche's text, you can learn about the construction of the eigenvariety from the original reference of Coleman and Mazur and from Bellaïche's paper on p-adic L-functions.

- R. F. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.
- J. Bellaïche. Critical p-adic L-functions. *Invent. Math.*, 189(1):1–60, 2012.

Finally, Bourbaki seminars are excellent places to look to get modern impressions of big results. In this case, Emerton wrote an excellent article that follows closely the same threads I spoke on. (Of course, I learned from this article!)

- M. Emerton. p-adic families of modular forms (after Hida, Coleman, and Mazur). *Astérisque*, (339):Exp. No. 1013, vii, 31–61, 2011. Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026.

Finally, I will reference the remarkable paper of Serre

- J.-P. Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972)*, pages 191–268. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.

In this work he explains how to build p-adic families of Eisenstein series and, at the same time, the p-adic zeta function.

Lecture 3: Galois representations over eigenvarieties

The third lecture focused on the variation of Galois representations and their application to the infinite fern of Gouvêa and Mazur. The original work of Gouvêa and Mazur, which continues to inspire, is explained in:

- B. Mazur. An "infinite fern" in the universal deformation space of Galois representations. *Collect. Math.*, 48(1-2):155–193, 1997. Journées Arithmétiques (Barcelona, 1995).
- F. Q. Gouvêa and B. Mazur. On the density of modular representations. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 127–142. Amer. Math. Soc., Providence, RI, 1998.

In the middle of the lecture, I explained how one actually proves or constructs the eigenfamilies. You can read this in Bellaïche's book, or Coleman and Mazur's paper, referenced in Lecture 2. You can also read portions of Coleman's paper on p-adic families, which Gouvêa and Mazur reference:

- R. F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

I also explained aspects of how the theory changes for $\mathrm{GL}(n)$. Hansen's reference in Lecture 2 is a good place to look, for the general story of any reductive group over the rationals, in fact. I also mentioned Urban's paper

- E. Urban. Eigenvarieties for reductive groups. *Ann. of Math. (2)*, 174(3):1685–1784, 2011.

There was a comment as to what eigenvariety Urban actually constructs — and it's true that there are some restrictions — but his paper is one place to learn about the actual objects (p-adic weights, distributions on Iwahori subgroups, and so on). In this vein I will also mention a really tangible papers of Ash, Pollack, and Stevens and, Calegari and Mazur, on the case of GL(3) over the rationals and GL(2) over an imaginary quadratic field, respectively:

- A. Ash, D. Pollack, and G. Stevens. Rigidity of p-adic cohomology classes of congruence subgroups of $\mathrm{GL}(n, \mathbb{Z})$. *Proc. Lond. Math. Soc. (3)*, 96(2):367–388, 2008.
- F. Calegari and B. Mazur. Nearly ordinary Galois deformations over arbitrary number fields. *J. Inst. Math. Jussieu*, 8(1):99–177, 2009.

1. Lecture IV: Pseudorepresentations

The final lecture focused on the algebra underlying the variation of Galois representations of eigenvarieties. The modern reference for pseudorepresentations is Chenevier's paper:

- G. Chenevier. The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 221–285. Cambridge Univ. Press, Cambridge, 2014.

The very most original papers on pseudorepresentations are a paper by Wiles and a paper by Taylor:

- A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.
- R. Taylor. Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.*, 63(2):281–332, 1991.

(See Lemma 2.2.3 in Wiles and the introduction in Taylor.)

There are also papers that *use* these in problems in modular forms and elliptic curves. Try looking at Wang-Erickson–Wake and Calegari–Specter:

- P. Wake and C. Wang-Erickson. Deformation conditions for pseudorepresenations. *Forum Math. Sigma*, 7:e20, 2019.
- F. Calegari and J. Specter. Pseudorepresentations of weight one are unramified. *Algebra Number Theory*, 13(7):1583–1596, 2019.

The argument of how to use the pseudorepresentation to build families of Galois representations is fundamentally due to Chenevier, in his paper of eigenvarieties for unitary groups compact at infinity:

- G. Chenevier. Familles p-adiques de formes automorphes pour GL_n. *J. Reine Angew. Math.*, 570:143–217, 2004.

Johansson and Newton explained how to adapt the argument to the case of $\mathrm{GL}(n)$ over $\mathbf{Q}$ in

- C. Johansson and J. Newton. Extended eigenvarieties for overconvergent cohomology. *Algebra Number Theory*, 13(1):93–158, 2019.

Finally, the results I know about infinite ferns beyond $\mathrm{GL}(2)$ are in papers of Chenevier and Hellmann–Margerin–Schraen:

- G. Chenevier. On the infinite fern of Galois representations of unitary type. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(6):963–1019, 2011.
- E. Hellmann, C. M. Margerin, and B. Schraen. Density of automorphic points in deformation rings of polarized global Galois representations. *To appear in Duke J. Math.*, 2022. Available at `arXiv:1811.09116`.

(These papers are challenging to read without learning a little more about p-adic automorphic forms on unitary groups compact at infinity.)

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