DAY 1: AFFINE AND PROJECTIVE VARIETIES

Throughout these notes $\mathbf{k}$ will denote an algebraically closed field (you may take $\mathbf{k} = \mathbb{C}$ if you feel more comfortable).

1. Affine varieties

We let $A = \mathbf{k}[x_1, \cdots, x_n]$ denote a polynomial ring, and consider an ideal $I \subseteq A$. The **algebraic set** defined by I is

$$V(I) = \{P \in \mathbf{k}^n : f(P) = 0 \text{ for all } f \in I\}. \tag{1.1}$$

If $X \subseteq \mathbf{k}^n$, we consider the **ideal of functions vanishing on** X

$$I(X) = \{f \in A : f(P) = 0 \text{ for all } P \in X\}. \tag{1.2}$$

We often write $\mathbb{A}^n$ for $\mathbf{k}^n$, and call it the **affine** n**-space**. You can check that $I(\mathbb{A}^n) = \langle 0 \rangle$!

An **algebraic variety** is an algebraic set $X \subseteq \mathbf{k}^n$ with the property that X is not expressible as

$$X = X_1 \cup X_2, \text{ where } X_1, X_2 \subsetneq X \text{ are strictly smaller algebraic sets.}$$

Equivalently, an algebraic set X is a variety if the ideal $I(X)$ is a prime ideal, which is in turn equivalent to the fact that the **affine coordinate ring**

$$A(X) = A/I(X)$$

is an integral domain (i.e. it contains no zero divisors). Note that

$$A(\mathbb{A}^n) = A/\langle 0 \rangle = A.$$

We will be interested in the case when X is one-dimensional, in which case we will call X an **affine curve**. The dimension of X is the same as that of its coordinate ring $A(X)$, and can be defined in terms of the length of ascending chains of prime ideals. For us the following criterion will be sufficient: if X is a variety then

$$\dim(X) = 0 \iff A(X) = \mathbf{k} \iff X \text{ is a single point}$$

$$\dim(X) = 1 \iff \mathbf{k} \subsetneq A(X) \text{ and for any (some) non-zero element } f \in A(X) \setminus \mathbf{k} \text{ we have that } A(X)/f \cdot A(X) \text{ is a finite dimensional vector space.}$$

$$\dim(X) \geq 2 \iff \mathbf{k} \subsetneq A(X) \text{ and for any (some) non-zero element } f \in A(X) \setminus \mathbf{k} \text{ we have that } A(X)/f \cdot A(X) \text{ is an infinite dimensional vector space.}$$

Based on this, one can check that $\dim(\mathbb{A}^1) = 1$, so $\mathbb{A}^1$ is a curve, called the **affine line**.

The zero-dimensional varieties in $\mathbb{A}^n$ are therefore just the points of $\mathbb{A}^n$, and we have that for every $P = (a_1, \cdots, a_n) \in \mathbb{A}^n$ the ideal $I(P)$ of functions vanishing at P is a maximal ideal in A

$$I(P) = (x_1 - a_1, \cdots, x_n - a_n).$$

Moreover, every maximal ideal in A has this form by the following.

Theorem 1.1 (Hilbert's Nullstelensatz)**.** *Every maximal ideal in $A = \mathbf{k}[x_1, \cdots, x_n]$ has the form (under the assumption that $\mathbf{k}$ is algebraically closed!)*

$$I(P) = (x_1 - a_1, \cdots, x_n - a_n) \text{ for some } P = (a_1, \cdots, a_n) \in \mathbb{A}^n.$$

Furthermore, if X is any variety then the prime ideal $I(X)$ is given by

$$I(X) = \bigcap_{P \in X} I(P).$$

Example 1.2. The ring $\mathbf{k}[x, y]/(y^2 - x^3)$ is the coordinate ring of the affine curve $C = V(y^2 - x^3) \subset \mathbb{A}^2$. To see that C is one-dimensional we consider the quotient

$$A(C)/xA(C) = \mathbf{k}[x, y]/(x, y^2 - x^3) = \mathbf{k}[y]/(y^2) = \mathbf{k} \oplus \mathbf{k} \cdot y$$

which is a 2-dimensional vector space. Our criterion then guarantees that $\dim(C) = 1$, so C is an affine curve. If instead we consider $X = V(y^2 - x^3) \subset \mathbb{A}^3$ then

$$A(X)/xA(X) = \mathbf{k}[x, y, z]/(x, y^2 - x^3) = \mathbf{k}[y, z]/(y^2)$$

which is infinite dimensional, hence X has dimension at least two (in fact exactly two).

2. Projective space and projective varieties

We define **projective n-space** over the field $\mathbf{k}$ to be the set of equivalence classes

$$\mathbb{P}^n = \frac{\mathbf{k}^{n+1} \setminus (0, \cdots, 0)}{\sim} \tag{2.1}$$

where the equivalence relation $\sim$ is defined via

$$(a_0, \cdots, a_n) \sim (\lambda \cdot a_0, \cdots, \lambda \cdot a_n) \text{ for all } 0 \neq \lambda \in \mathbf{k} \text{ and } (a_0, \cdots, a_n) \in \mathbf{k}^{n+1} \setminus 0.$$

We write $[a_0 : a_1 : \cdots : a_n]$ for the equivalence class of $(a_0, \cdots, a_n)$, and we think of the points in $\mathbb{P}^n$ as parametrizing the lines through the origin in $\mathbf{k}^{n+1}$.

We let $S = \mathbf{k}[X_0, \cdots, X_n]$ and consider the grading on S defined by

$$S_d = \{\text{homogeneous polynomials of degree } d\} = \bigoplus_{i_0 + \cdots + i_n = d} \mathbf{k} \cdot X_0^{i_0} \cdots X_n^{i_n}.$$

Note that if $F \in S_d$ then

$$F(\lambda \cdot a_0, \cdots, \lambda \cdot a_n) = \lambda^d \cdot F(a_0, \cdots, a_n),$$

so despite the fact that it doesn't make sense to evaluate F at a point $P \in \mathbb{P}^n$, one can make sense of whether F **vanishes** at P or not. We can thus define

$$V(I) = \{P \in \mathbb{P}^n : F(P) = 0 \text{ for all homogeneous polynomials } F \in I\} \tag{2.2}$$

the **projective algebraic set** defined by an ideal I. Even though the above definition makes sense for arbitrary ideals, we will only use it in the case when I is a **homogeneous ideal**, i.e. I is generated by homogeneous polynomials. Moreover, given a subset $X \subseteq \mathbb{P}^n$ we let

$$I(X) = \text{ the ideal generated by } \{F \in S \text{ homogeneous} : F(P) = 0 \text{ for all } P \in X\}. \tag{2.3}$$

We consider the homogeneous coordinate ring of a projective algebraic set $X \subseteq \mathbb{P}^n$ to be

$$S(X) = S/I(X) \tag{2.4}$$

X is a **projective variety** if $I(X)$ is a prime ideal, or equivalently $S(X)$ is a domain.

2.1. **Exercises.**

(1)
- Show that the set
$$C = \{(t^2, t^3) : t \in \mathbf{k}\}$$
is the same as the affine algebraic set
$$V(y^2 - x^3).$$
- Show then that
$$I(C) = \langle y^2 - x^3 \rangle \subset \mathbf{k}[x, y],$$
and conclude that C is an affine algebraic variety.
- Show that $\dim(C) = 1$ (C is called the **cuspidal cubic curve**).

(2) If X, Y are affine algebraic sets in $\mathbf{k}^n$ and I, J are ideals in A, show that
- If $X \subseteq Y$ then $I(X) \supseteq I(Y)$.
- If $I \subseteq J$ then $V(I) \supseteq V(J)$.
- If we let $\sqrt{I} = \{a \in A : a^r \in I \text{ for some } r > 0\}$ denote the **radical** of the ideal I then
$$V(I) = V(\sqrt{I}).$$
- Using the fact that
$$\sqrt{I} = \bigcap_{\substack{\mathfrak{P} \supseteq I \\ \mathfrak{P} \text{ prime ideal}}} \mathfrak{P}$$
and Theorem 1.1 show that for every ideal $I \subset A$ we have
$$I(V(I)) = \sqrt{I}.$$

(3) Show that for any two ideals $I, J \subseteq A$ we have
$$V(I \cdot J) = V(I \cap J) = V(I) \cup V(J).$$
Prove that if X is an algebraic set then we have an equivalence
$$X \text{ is irreducible} \iff I(X) \text{ is a prime ideal.}$$

(4) State and prove the analogous statements in Exercises 2 and 3 for homogeneous ideals and projective algebraic sets/varieties.

(5) Verify that if $X = \{[a_0 : \cdots : a_n]\}$ consists of a single point in $\mathbb{P}^n$ then the homogeneous ideal of X is

$$I(X) = \langle a_i x_j - a_j x_i : 0 \leq i < j \leq n \rangle \subset S = \mathbf{k}[x_0, \cdots, x_n],$$

and that the homogeneous coordinate ring of X is isomorphic to a polynomial ring in one variable.

By looking at specific points in $\mathbb{P}^n$, $n = 1, 2, 3, \cdots$, convince yourself that in general $I(X)$ can be generated by only n linear homogeneous polynomials (so the $\binom{n+1}{2}$ polynomials that I wrote down give in general a reduntant set of generators for $I(X)$).

(6) Verify that you can decompose

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1},$$

where $\mathbb{A}^n \cong \{[1 : a_1 : \cdots : a_n]\}$ and $\mathbb{P}^{n-1} \cong \{[0 : a_1 : \cdots : a_n]\}$. We call $\mathbb{P}^{n-1}$ the **hyperplane at infinity**, parametrizing the directions of the lines through the origin in $\mathbb{A}^n$.

In fact, if we let

$$U_i = \{[a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n]\} \subset \mathbb{P}^n, \quad i = 0, \cdots, n,$$

then we can naturally identify U_i with $\mathbb{A}^n$, and its complement $H_i = \mathbb{P}^n \setminus U_i$ with $\mathbb{P}^{n-1}$.

(7) Consider the subset

$$X = \{[s^3 : st^2 : t^3] : s, t \in \mathbf{k}, \text{ not both } s, t \text{ are } 0\} \subset \mathbb{P}^2 = \{[w : x : y]\}.$$

Show that

$$I(X) = \langle y^2 w - x^3 \rangle \subset \mathbf{k}[w, x, y],$$

and conclude that X is a projective variety. Using the notation in Exercise 6, show that $X \cap U_0$ is naturally identified with the cuspidal curve C in $\mathbb{A}^2 = U_0$, and that

$$X = C \cup \{[0 : 0 : 1]\},$$

where you can think of $[0 : 0 : 1]$ as the **point at infinity** on the cuspidal curve. Note also that $y^2 w - x^3$ is the **homogenization** of the equation $y^2 - x^3$ of C, with respect to the variable w.