# DAY 2: INVARIANTS OF ALGEBRAIC VARIETIES

## 1. Hilbert function and polynomial for projective varieties

Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set. The **Hilbert function of** $X$ is the function $\mathrm{HF}_X : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$\mathrm{HF}_X(k) = \dim_{\mathbf{k}} S(X)_k.$$

**Theorem 1.1.** *There exists a univariate polynomial* $\mathrm{HP}_X(z) \in \mathbb{Q}[z]$ *with the property that*

$$\mathrm{HF}_X(k) = \mathrm{HP}_X(k) \text{ for } k \gg 0.$$

*The polynomial* $\mathrm{HP}_X(z)$ *is called the* **Hilbert polynomial of** $X$*.*

The degree of the polynomial $\mathrm{HP}_X(z)$ measures the **dimension of** $X$. If we write $r = \dim(X)$ then

$$\mathrm{HP}_X(z) = a_r \cdot z^r + \text{ lower order terms}$$

We define

$$e(X) = a_r \cdot r! \text{ to be the } \textbf{degree} \text{ of the algebraic set } X. \tag{1.1}$$

A **projective curve** is a projective algebraic variety $C \subseteq \mathbb{P}^n$ (i.e. the ideal $I(C)$ is a homogeneous prime ideal) of dimension one (so that the Hilbert polynomial is of the form $\mathrm{HP}_C(z) = e \cdot z + c$, where $e = e(C)$ is the degree of the curve).

**Example 1.2** (The projective line)**.** If $C = \mathbb{P}^1$ then the coordinate ring of $C$ is

$$S(C) = \mathbf{k}[X_0, X_1],$$

the Hilbert function is

$$\mathrm{HF}_C(k) = k + 1,$$

the Hilbert polynomial is

$$\mathrm{HP}_C(z) = z + 1$$

so the dimension is $\dim(C) = 1$, and the degree is $\deg(C) = 1$. Since $I(C) = 0$ is a prime ideal, $C$ is a curve.

**Example 1.3** (Degree $d$ plane curves)**.** Let $S = \mathbf{k}[X_0, X_1, X_2]$, $d > 0$, and let $F \in S_d$ be an irreducible polynomial of degree $d$. Let $C = V(F) \subset \mathbb{P}^2$ and note that

$$I(C) = F \cdot S \text{ is a prime ideal, } S(C) = S/F \cdot S \text{ is the coordinate ring of } C,$$

the Hilbert function of $C$ is given by

$$\mathrm{HF}_C(k) = \begin{cases} \binom{k+2}{2} & \text{if } k < d; \\ \frac{d\cdot(2k+3-d)}{2} & \text{if } k \geq d. \end{cases}$$

The Hilbert polynomial of $C$ is given by

$$\mathrm{HP}_C(z) = d \cdot z - \frac{d(d-3)}{2}, \tag{1.2}$$

so the dimension of $C$ is $\dim(C) = 1$ and the degree of $C$ is $\deg(C) = d$.

## 2. Function field, local rings

Let $X \subseteq \mathbf{k}^n$ be an affine variety. The **function field of** $X$ is the fraction field of its affine coordinate ring:

$$K(X) = K^{aff}(X) = \mathrm{Frac}(A(X)) = \left\{ \frac{f}{g} : f, g \in A(X), g \neq 0 \right\}.$$

One can think of $K(X)$ as functions on $X$ which are **partially defined** (away from the locus when $g = 0$): we call them **rational functions**, or **meromorphic functions**. Given a point $P \in X$ we define the **local ring of** $X$ **at** $P$ by

$$\mathcal{O}_P = \mathcal{O}_{X,P} = \left\{ \frac{f}{g} \in K(X) : g(P) \neq 0 \right\}.$$

In other words, $\mathcal{O}_P$ is the ring of meromorphic functions which are defined at the point $P$ (we say that they are **regular at** $P$)! It is then not surprising that

$$A(X) = \bigcap_{P \in X} \mathcal{O}_P \tag{2.1}$$

where the intersection takes place in $K(X)$.

Consider now a projective variety $X \subseteq \mathbb{P}^n$. The **function field of** $X$ is defined by

$$K(X) = K^{proj}(X) = \{0\} \cup \left\{ \frac{F}{G} : 0 \neq F, G \in S(X)_k \text{ for some } k \geq 0 \right\}.$$

Note that $F, G$ are not functions, but their ratio is (at least away from the locus when $G = 0$)! If $P \in X$ we define the **local ring of** $X$ **at** $P$ by

$$\mathcal{O}_P = \mathcal{O}_{X,P} = \{0\} \cup \left\{ \frac{F}{G} \in K(X) : G(P) \neq 0 \right\}.$$

We use the same terminology as in the affine case: elements of $K(X)$ are called rational/meromorphic functions, and the ones in $\mathcal{O}_P$ are regular at $P$. In contrast with (2.1), when $X$ is a projective variety we have

$$\mathbf{k} = \bigcap_{P \in X} \mathcal{O}_P \tag{2.2}$$

which means that the only rational functions that are everywhere regular are the constants!

## 3. Order function, intersection multiplicity

Let $R$ be a $\mathbf{k}$-algebra satisfying

- $R$ is a local ring with maximal ideal $\mathfrak{m}$ and the inclusion of $\mathbf{k} \subset R$ induces an isomorphism
$$\mathbf{k} \simeq R/\mathfrak{m}.$$
- $R$ is a domain.
- $R$ is one dimensional, i.e. for every non-zero $a \in R$ we have that $R/aR$ is a finite dimensional vector space (equivalently, the only prime ideals in $R$ are $(0)$ and $\mathfrak{m}$).

The main example of rings satisfying the conditions above that will concern us are local rings $R = \mathcal{O}_P$ at points $P$ on an affine or projective curves.

For every $0 \neq a \in R$ we define the **order** of $a$ with respect to $R$ to be

$$\operatorname{ord}(a) = \operatorname{ord}_R(a) = \dim_{\mathbf{k}}(R/aR)$$

More generally, consider $K = \operatorname{Frac}(R)$ the fraction field of $R$, and let $f \in K$. We define the order of $f$ with respect to $R$ to be the (possibly negative) integer

$$\operatorname{ord}(f) = \operatorname{ord}_R(f) = \operatorname{ord}(a) - \operatorname{ord}(b) \text{ for any expression } f = \frac{a}{b} \text{ with } a, b \in R. \tag{3.1}$$

It can be shown that the above definition is independent of the representation of $f$ as a fraction $a/b$. We will apply the above definition with $R = \mathcal{O}_P$ the local ring of a curve $C$ at a point $P$, in which case $\mathfrak{m}_P \subseteq \mathcal{O}_P$ is the unique maximal ideal (see Exercise 2).

Suppose that $C_1, C_2 \subset \mathbf{k}^2$ are two (distinct) affine plane curves passing through the same point $P$, and let $C_i = V(f_i)$ for some polynomials $f_1, f_2$. We define the **intersection multiplicity** of $C_1$ and $C_2$ at $P$ to be

$$i_P(C_1, C_2) = \dim_{\mathbf{k}} \left( \mathcal{O}_{\mathbf{k}^2, P}/(f_1, f_2) \right), \tag{3.2}$$

and we have

$$\sum_{P \in C_1 \cap C_2} i_P(C_1, C_2) = \dim_{\mathbf{k}} \left( \mathbf{k}[x, y]/(f_1, f_2) \right)$$

For example, take $P = (0,0)$, $f_1 = y^2 - x^3$, $f_2 = x$. Since $P$ is the only intersection point of $V(f_1)$ and $V(f_2)$ we get

$$i_P(C_1, C_2) = \dim_{\mathbf{k}} \mathbf{k}[x,y]/(y^2 - x^3, x) = 2.$$

If instead we take $f_2 = y$ then we get

$$i_P(C_1, C_2) = \dim_{\mathbf{k}} \mathbf{k}[x,y]/(y^2 - x^3, y) = 3.$$

Suppose now that $C_1, C_2 \subset \mathbb{P}^2$ are projective plane curves passing through the same point $P$, and let $C_i = V(F_i)$ for some homogeneous polynomials $F_1, F_2$ of degrees $d_1, d_2$ respectively. We define the **intersection multiplicity** of $C_1$ and $C_2$ at $P$ as follows. If $P = [1:0:0]$ and if we dehomogenize $F_1, F_2$ by setting $f_1 = F_1(1, x_1, x_2)$, $f_2 = F_2(1, x_1, x_2)$, then the intersection multiplicity can be computed via (3.2). If $P \neq [1:0:0]$ then we first make a linear change of coordinates to move $P$ to $[1:0:0]$ and then use the above formula. An equivalent formulation of the above definition which avoids the change of coordinates goes as follows: let $L \in \mathbf{k}[X_0, X_1, X_2]_1$ denote a linear form which does not pass through $P$ (i.e. $L(P) \neq 0$), consider the local ring $\mathcal{O}_{\mathbb{P}^2,P}$ and the elements

$$f_i = \frac{F_i}{L^{d_i}} \in \mathcal{O}_{\mathbb{P}^2,P}.$$

The intersection multiplicity is then computed via

$$i_P(C_1, C_2) = \dim_{\mathbf{k}} \left( \frac{\mathcal{O}_{\mathbb{P}^2,P}}{(f_1, f_2)} \right).$$

The following theorem asserts that two projective plane curves of degrees $d_1$ and $d_2$ intersect in $d_1 \cdot d_2$ points, if the intersection points are counted with appropriate multiplicities (in particular there are no parallel lines!).

**Theorem 3.1** (Bézout)**.** *Let $S = \mathbf{k}[x_0, x_1, x_2]$, let $d_1, d_2 > 0$, let $F_i \in S_{d_i}$ be irreducible homogeneous polynomials, and consider the corresponding plane curves $C_i = V(F_i)$. We have*

$$\sum_{P \in C_1 \cap C_2} i_P(C_1, C_2) = d_1 \cdot d_2.$$

3.1. **Exercises.**

(1) Let $S = \mathbf{k}[x_0, x_1, x_2]$ and consider irreducible homogeneous polynomials $F_1 \in S_{d_1}$, $F_2 \in S_{d_2}$, which are not scalar multiples of each other. Consider the corresponding curves $C_i = V(F_i)$, and the ideal $I = \langle F_1, F_2 \rangle$. The goal of this exercise is to compute the Hilbert function of $S/I$, defined by

$$HF_{S/I}(m) = \dim_{\mathbf{k}}(S/I)_m, \quad m \geq 0.$$

- Consider first the homogeneous coordinate ring $S(C_1) = S/\langle F_1 \rangle$, and verify the calculation in Example 1.3:

$$HF_{C_1}(m) = \dim(S(C_1)_m) = \begin{cases} \binom{m+2}{2} & \text{if } m < d_1; \\ \frac{d_1 \cdot (2m+3-d_1)}{2} & \text{if } m \geq d_1. \end{cases}.$$

- Explain why multiplication by $F_2$ is injective on $S(C_1)$, more precisely, why it gives injective $\mathbf{k}$-linear maps

$$S(C_1)_{m-d_2} \xrightarrow{\cdot F_2} S(C_1)_m \text{ for all } m.$$

- Use the identification $S/I \cong S(C_1)/\langle F_2 \rangle$ to conclude that

$$HF_{S/I}(m) = HF_{C_1}(m) - HF_{C_1}(m - d_2) \text{ for all } m,$$

  and write down an explicit formula for each $m$ (observe the symmetry between $d_1$ and $d_2$).
- Verify that $HF_{S/I}(m) = d_1 d_2$ for $m \gg 0$, which is another formulation of Bézout's theorem.

(2) (a) If $X$ is an affine variety and $P \in X$, check that $\mathcal{O}_P \subset K^{aff}(X)$ has a unique maximal ideal, namely

$$\mathfrak{m}_P = \{f/g : f(P) = 0, g(P) \neq 0\}.$$

(b) If $X \subseteq \mathbb{P}^n$ is a projective variety, check that the set $K^{proj}(X)$ defined in Section 2 is indeed a field. Prove that the subring $\mathcal{O}_P \subset K(X)$ has a unique maximal ideal, namely

$$\mathfrak{m}_P = \{F/G : F(P) = 0, G(P) \neq 0\}.$$

(c) Using Hilbert's Nullstellensatz, check that for both affine and projective varieties, if $P \in X$ and $\mathcal{O}_P$ is the corresponding local ring with maximal ideal $\mathfrak{m}_P$, then the natural inclusion $\mathbf{k} \subset \mathcal{O}_P$ induces an isomorphism $\mathbf{k} \simeq \mathcal{O}_P/\mathfrak{m}_P$.

(d) Prove (2.1).

(3) Consider the cuspidal cubic $C = V(y^2 - x^3) \subset \mathbb{A}^2$.

- Show that $P = (1, 1)$ belongs to $C$, and that the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_P$ is **principal**, that is, it can be generated by a single element. More precisely, show that $\mathfrak{m}_P = \langle x - 1 \rangle$ and also $\mathfrak{m}_P = \langle y - 1 \rangle$.
- By contrast, convince yourself that for $P = (0, 0)$, the ideal $\mathfrak{m}_P$ is NOT principal.

(4) Consider the projective plane curves $C_1, C_2 \subset \mathbb{P}^2$,

$$C_1 = V(X^2 + Y^2 - Z^2) \text{ and } C_2 = V(X^3 - X^2Z - XZ^2 + Z^3 - Y^2Z).$$

Determine the set $C_1 \cap C_2$ and find the intersection multiplicity at each of these points. Verify that Bézout's theorem holds in this example.

(5) For a homogeneous polynomial $F$ of degree $d$ prove the Euler identity

$$\sum_{i=0}^{n} X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F.$$