1. Divisors and linear equivalence

Throughout the rest of the lectures we will assume that C is a projective curve. A **divisor** on C is a formal linear combination

$$D = \sum_{P \in C} a_P \cdot P$$

where only finitely many coefficients a_P are non-zero. Divisors on C form a group, denoted $\mathrm{Div}(C)$, which is a free abelian group on the set of points of C. We say that a divisor D is **effective**, and write $D \geq 0$, if $a_P \geq 0$ for all $P \in C$. More generally, we write $D \geq E$ if $D - E$ is effective. We define the **degree** of a divisor D to be

$$\deg(D) = \sum_{P \in C} a_P.$$

Let $C \subseteq \mathbb{P}^n$ be a curve and $F \in \kappa[x_0, \cdots, x_n]$ a homogeneous polynomial of degree d such that $C \not\subseteq V(F)$. There is natural generalization of Bézout's theorem which can be formulated as follows. Given a point $P \in C$ with $F(P) = 0$ we consider any linear form L with $L(P) \neq 0$ and write

$$f = \frac{F}{L^d} \in \mathcal{O}_{C,P}, \quad \text{and} \quad i_P(C, F) = \dim_\kappa \left(\frac{\mathcal{O}_{C,P}}{(f)} \right).$$

With the definition of the degree $e(C)$ of the curve C from the previous lecture, we have

$$\sum_{P \in C \cap V(F)} i_P(C, F) = e(C) \cdot d. \tag{1.1}$$

We can then associate a natural divisor on C to F via

$$\mathrm{Div}(F) = \sum_{P \in C \cap V(F)} i_P(C, F) \cdot P, \tag{1.2}$$

which is effective, and satisfies

$$\deg(\mathrm{Div}(F)) = e(C) \cdot \deg(F).$$

In the special case when C is a plane curve of degree d and F is a linear form, it follows that $\mathrm{Div}(F)$ is an effective divisor of degree d.

If $f \in K(C)$ is a non-zero rational function on C, and if $\mathcal{O}_P$ is the local ring of C at a point P, we define $\mathrm{ord}_P(f)$ to be the order of f with respect to the local ring $\mathcal{O}_P$. We will assume throughout this section that C is a **non-singular** curve, i.e. that the maximal ideal $\mathfrak{m}_P$ in each local ring $\mathcal{O}_P$ is a principal ideal, generated by some uniformizer π_P. Recall that in this case for every rational function $f \in K(C)$ and every point $P \in C$ we can write

$$f = u \cdot \pi_P^{\mathrm{ord}_P(f)}, \text{ where } u \in \mathcal{O}_P \text{ is a unit.}$$

If $\operatorname{ord}_P(f) > 0$ then we say that P is a **zero of** f, while if $\operatorname{ord}_P(f) < 0$ then P is a **pole of** f.

The **principal divisor** associated to a rational function f is (not to be confused with (1.2)!!!)

$$\operatorname{Div}(f) = \sum_{P \in C} \operatorname{ord}_P(f) \cdot P, \tag{1.3}$$

which in turn can be written as a difference $\operatorname{Div}(f) = \operatorname{Div}_0(f) - \operatorname{Div}_\infty(f)$ of effective divisors, where

$$\operatorname{Div}_0(f) = \sum_{\operatorname{ord}_P(f)>0} \operatorname{ord}_P(f) \cdot P \text{ is the **divisor of zeroes** of } f, \text{ and}$$

$$\operatorname{Div}_\infty(f) = \sum_{\operatorname{ord}_P(f)<0} (-\operatorname{ord}_P(f)) \cdot P \text{ is the **divisor of poles** of } f.$$

Alternatively, if $f = F/G$ where F, G are homogeneous of the same degree then

$$\operatorname{Div}(f) = \operatorname{Div}(F) - \operatorname{Div}(G),$$

where the left hand side is as defined in (1.3), while the right hand side is defined by (1.2).

Theorem 1.1. *Every principal divisor has degree* 0. *In other words, every rational function on a curve has the same number of zeroes and poles, if they are counted appropriately! Moreover, if* f *is a non-constant rational function then* f *has at least one zero and at least one pole.*

We write $\operatorname{Div}(C)$ for the group of divisors on C, and $\operatorname{PDiv}(C) \subset \operatorname{Div}(C)$ for the subgroup of principal divisors. We let

$$\operatorname{Cl}(C) = \frac{\operatorname{Div}(C)}{\operatorname{PDiv}(C)} \text{ the **class group** of } C,$$

which is the same as the group of equivalence classes of divisors for the equivalence relation defined by $D \equiv D'$ if and only if $D' - D$ is a principal divisor. If $D \equiv D'$ then we say that D and D' are **linearly equivalent**. The function that measures the degree of a divisor gives rise to a surjective group homomorphism $\deg : \operatorname{Div}(C) \longrightarrow \mathbb{Z}$. Theorem 1.1 implies that linearly equivalent divisors have the same degree, hence this function factors through the class group:

$$\deg : \operatorname{Cl}(C) \longrightarrow \mathbb{Z}.$$

We let

$$\operatorname{Cl}^0(C) = \{D \in \operatorname{Cl}(C) : \deg(D) = 0\}.$$

In Exercise 1 you will verify that $\operatorname{Cl}^0(\mathbb{P}^1) = 0$, while in the case of plane cubic curves E (which have a group structure as explained in Evan's lecture) we have $\operatorname{Cl}^0(E) \simeq E$ via the isomorphism described in the following theorem, whose proof will be given as a consequence of the Riemann-Roch formula.

Theorem 1.2. *Suppose that $E \subseteq \mathbb{P}^2$ is a non-singular cubic curve and fix a point $O \in E$. We have a bijection*

$$\phi : E \longrightarrow \mathrm{Cl}^0(E), \quad \phi(P) = P - O.$$

Since $\mathrm{Cl}^0(E)$ is naturally a group, we can use ϕ^{-1} to transport the group structure on E: this agrees with the geometric construction in Evan's lecture.

To a divisor D on the curve C we associate the vector space

$$L(D) = \{f \in K(C) : \mathrm{Div}(f) + D \geq 0\}, \tag{1.4}$$

and write $l(D) = \dim_\kappa(L(D))$. If $D = \sum a_P \cdot P$ then the condition that $f \in L(D)$ is equivalent to

$$\mathrm{ord}_P(f) \geq a_P \text{ for all } P \in C.$$

Proposition 1.3. *Let D, D' be divisors on a nonsingular curve C. We have*

(1) *If $D \leq D'$ then $L(D) \subseteq L(D')$ and*

$$\dim_\kappa \frac{L(D')}{L(D)} \leq \deg(D' - D).$$

(2) *$L(0) = \kappa$ and $L(D) = 0$ if $\deg(D) < 0$.*

(3) *$L(D)$ is finite dimensional and $l(D) \leq \deg(D) + 1$.*

(4) *If $D \equiv D'$ then $l(D) = l(D')$.*

Proof. If $D \leq D'$ then it is clear that $L(D) \subseteq L(D')$. To prove the upper bound on the dimension of $L(D')/L(D)$ it is sufficient to verify the case when $D' = D + P$ for some point $P \in C$. Let a be the coefficient of P in D, so that $\mathrm{ord}_P(f) \geq -(a+1)$ for all $f \in L(D')$. Let π be a uniformizer at the point P and consider the κ-linear map

$$\phi : L(D') \longrightarrow \kappa, \quad \phi(f) = (f \cdot \pi^{a+1})(P),$$

which is well-defined since $\mathrm{ord}_P(f \cdot \pi^{a+1}) \geq 0$ for all $f \in L(D')$. The kernel of ϕ is $L(D)$, while the image is at most one dimensional. This shows

$$\dim_\kappa \frac{L(D')}{L(D)} \leq 1 = \deg(D' - D),$$

proving (1). To prove (2), note that every constant function is in $L(0)$. Moreover, if $f \in L(0)$ then $\mathrm{Div}(f) \geq 0$ so f has no poles, which by Theorem 1.1 means that f has to be constant. If $\deg(D) < 0$ and $f \in L(D)$ then $\deg(\mathrm{div}(f) + D) \geq 0$, but since $\deg(\mathrm{Div}(f)) = 0$ this implies $\deg(D) \geq 0$ which is a contradiction.

Conclusion (4) follows from an explicit isomorphism between $L(D)$ and $L(D')$ for $D' - D = \mathrm{Div}(h)$, namely:

$$\psi : L(D) \longrightarrow L(D'), \quad \psi(f) = f \cdot h.$$

Finally, to prove (3) it suffices to replace D by a linearly equivalent divisor (since both sides of the inequality are unchanged in view of (4) and Theorem 1.1). If $l(D) = 0$ then the inequality is trivial. Otherwise, let $0 \neq f \in L(D)$ and define $D' = D + \mathrm{Div}(f) \geq 0$. Using (1) and (2) yields

$$l(D') - 1 = \dim_\kappa \frac{L(D')}{L(D)} \leq \deg(D' - 0) = \deg(D'),$$

and since $D' \equiv D$ the conclusion follows. □

Proposition 1.4. *Let $P_1, \cdots, P_m \in C$ and let $a_1, \cdots, a_m \in \mathbb{Z}$. There exists a rational function $f \in K(C)$ with*

$$\mathrm{ord}_{P_i}(f) = a_i \text{ for all } i = 1, \cdots, m.$$

Proof. It is enough to consider the case when $a_i = 1$ and $a_j = 0$ for $j \neq i$. Indeed, if we can find f_i with $\mathrm{ord}_{P_i}(f_i) = 1$ and $\mathrm{ord}_{P_j}(f_j) = 0$ for $j \neq i$ then for arbitrary $a_1, \cdots, a_m \in \mathbb{Z}$ we can take $f = f_1^{a_1} \cdots f_m^{a_m}$.

Up to a change of coordinates, we may assume that all the points P_i lie in the affine space $\kappa^n = U_0 = (x_0 \neq 0)$ and restrict our attention to the affine curve $C_0 = C \cap U_0$. We let $R = A(C_0)$ denote the affine coordinate ring of C_0, and let $I_i = I(P_i) = \{f \in R : \mathrm{ord}_{P_i}(f) \geq 1\}$ denote the maximal ideals of R corresponding to the points P_i.

Note that every element $f \in R \subset K(C)$ satisfies $\mathrm{ord}_P(f) \geq 0$ for all $P \in C_0$ and consider the ideal

$$I_{2P_i} = \{f \in R : \mathrm{ord}_{P_i}(f) \geq 2\}.$$

It follows that the elements f in R with the property that $\mathrm{ord}_{P_i}(f) = 1$ and $\mathrm{ord}_{P_j}(f) = 0$ for $j \neq i$ are precisely the elements in

$$I_{P_i} \setminus (I_1 \cup \cdots \cup I_{i-1} \cup I_{2P_i} \cup I_{i+1} \cup \cdots \cup I_r)$$

If no such element exists then by Exercise 5 and the fact that $I_i \not\subset I_j$ for $j \neq i$, we must have $I_i \subseteq I_{2P_i}$, but this is not the case: choose π_i an uniformizer at P_i and write it as $\pi_i = f_i/g_i$ with $f_i, g_i \in R$ and $g_i(P_i) \neq 0$. It follows that $\mathrm{ord}_{P_i}(g_i) = 0$ and

$$\mathrm{ord}_{P_i}(f_i) = \mathrm{ord}_{P_i}(\pi_i) = 1.$$

This means $f_i \in I_{P_i} \setminus I_{2P_i}$, so the inclusion $I_{P_i} \subseteq I_{2P_i}$ cannot be true. □

It will be useful to define the following variation of the vector space $L(D)$ in (1.4). Let $\mathcal{S} \subseteq C$ be any subset and for $D = \sum a_P \cdot P$ define

$$L^{\mathcal{S}}(D) = \{f \in K(C) : \mathrm{ord}_P(f) \geq -a_P \text{ for all } P \in \mathcal{S}\}, \tag{1.5}$$

and let $l^{\mathcal{S}}(D) = \dim_\kappa L^{\mathcal{S}}(D)$, and $\deg^{\mathcal{S}}(D) = \sum_{P \in \mathcal{S}} a_P$. Note that when $\mathcal{S} = C$ we recover the previous definition: $L^C(D) = L(D)$ and $l^C(D) = l(D)$. In Exercise 6 you will prove the following version of Proposition 1.3(1).

Lemma 1.5. *If $D \leq D'$ and $\mathcal{S} \subseteq C$ is any subset then $L^{\mathcal{S}}(D) \subseteq L^{\mathcal{S}}(D')$. If $\mathcal{S}$ is finite then*

$$\dim_\kappa \frac{L^{\mathcal{S}}(D')}{L^{\mathcal{S}}(D)} = \deg^{\mathcal{S}}(D' - D). \tag{1.6}$$

The next result will be essential in the proof of Riemann's theorem next time.

Proposition 1.6. *Let $f \in K(C) \setminus \kappa$ be a non-constant rational function, and let $n = [K(C) : \kappa(f)]$ denote the degree of the finite field extension $\kappa(f) \subseteq K(C)$. We have*

1. *$\mathrm{Div}_0(f)$ is an effective divisor of degree n.*
2. *There exists a constant τ such that*

$$l(r \cdot \mathrm{Div}_0(f)) \geq r \cdot n - \tau \text{ for all } r.$$

Proof of part (1). Let $Z = \mathrm{Div}_0(f) = \sum a_P \cdot P$ and let $m = \deg(f)$: we first show that $m \leq n$. Consider

$$\mathcal{S} = \{P : a_P > 0\}$$

and choose $f_1, \cdots, f_m \in L^{\mathcal{S}}(0)$ with the property that their residues $\overline{f_i}$ in the quotient

$$\frac{L^{\mathcal{S}}(0)}{L^{\mathcal{S}}(-Z)}$$

form a basis. To prove $m \leq n$ we need to check that $f_1, \cdots, f_m$ are linearly independent over $\kappa(f)$: if not then there exist polynomials $g_i(T) \in \kappa[T]$ with the property that

$$\sum_i g_i(f) \cdot f_i = 0$$

and we may assume that some g_i has a non-zero constant term. If we write $g_i(T) = c_i + T \cdot h_i(T)$, where $c_i \in \kappa$ then we get

$$\sum_i c_i \cdot f_i = -f \cdot \sum_i h_i(f) \cdot f_i \in L^{\mathcal{S}}(-Z)$$

Working in the quotient $\frac{L^{\mathcal{S}}(0)}{L^{\mathcal{S}}(-Z)}$ the above relation implies that $\overline{f_i}$ are linearly dependent, which is a contradiction. Note that the inequality $m \geq n$ follows from (2):

$$r \cdot m + 1 = r \cdot \deg(\mathrm{Div}_0(f)) + 1 \overset{\text{Proposition } 1.3}{\geq} l(r \cdot \mathrm{Div}_0(f)) \geq r \cdot n - \tau$$

so by taking $r \gg 0$ we get $m \geq n$. □

Part (2) is more technical, but you can see some examples in the exercises.

1.1. **Exercises.**

(1) Let $C = \mathbb{P}^1$ with projective coordinates $[X : Y]$, and consider the rational function

$$t = Y/X \in K(\mathbb{P}^1).$$

(a) Show that $K(\mathbb{P}^1) = \kappa(t)$, and compute the principal divisor $\mathrm{Div}(t)$.

(b) Show that if $E = \sum_P a_P \cdot P$ is an effective divisor on $\mathbb{P}^1$ of degree d, then there exists a homogeneous polynomial F of degree d with $\mathrm{Div}(F) = E$.

(c) Explain how to compute the divisor of an arbitrary rational function on $\mathbb{P}^1$.

(d) Show that $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$ and $\mathrm{Cl}^0(\mathbb{P}^1) = 0$.

(e) For the rational function $t = Y/X$, determine $l(r \cdot \mathrm{Div}_0(t))$ for all $r \geq 0$.

(f) Verify that Proposition 1.6 holds for $C = \mathbb{P}^1$.

(2) Fix a constant $\lambda \in \kappa$ with $\lambda \neq 0, 1$ and consider the cubic $E \subset \mathbb{P}^2$ defined by the equation

$$Y^2 Z = X \cdot (X - Z) \cdot (X - \lambda Z).$$

Consider the rational functions $x = X/Z$ and $y = Y/Z$.

(a) Show that $K(C) = \kappa(x, y)$ (where x, y satisfy the relation $y^2 = x(x-1)(x-\lambda)$).

(b) Determine $\mathrm{Div}(x)$ and $\mathrm{Div}(y)$.

(c) Show that if you let $z = x^{-1}$ then $L(\mathrm{Div}_0(z)) \subset \kappa[x, y]$ and prove that

$$l(r \cdot \mathrm{Div}_0(z)) = 2r \text{ for all } r > 0.$$

(3) We say (see the notes) that D and D' are **linearly equivalent** if $D' - D = \mathrm{Div}(h)$ is a principal divisor, and write $D \equiv D'$. Show the following:

- $l(D) > 0 \iff D$ is linearly equivalent to an effective divisor.
- $\deg(D) = 0$ and $l(D) > 0 \iff D \equiv 0 \iff D$ is a principal divisor.

(4) In this exercise you will prove (you can also look at the notes to see the details) that if $D \leq D'$ are divisors on a curve C then

$$l(D') \leq l(D) + \deg(D' - D).$$

- Explain why you can reduce to the case when $D' - D = P$ consists of a single point.
- Let a be the coefficient of P in D (so that $(a + 1)$ is the coefficient of P in D'), and let π denote the uniformizer at P (so that $\mathfrak{m}_P = \langle \pi \rangle$). Explain why the map

$$\phi : L(D') \longrightarrow \kappa, \quad \phi(f) = (f \cdot \pi^{a+1})(P),$$

defined by multiplication by π^{a+1} followed by evaluation at P, is κ-linear and is well-defined.

- Show that $L(D)$ is contained in $\ker(\phi)$ and deduce that $l(D') \leq l(D) + 1$.

(5) (**Prime avoidance**) Suppose that $J, Q_0, Q_1, \cdots, Q_r$ are ideals in a ring R, and assume further that $Q_1, \cdots, Q_r$ are prime ideals (Q_0 may not be prime). Show that if we have

$$J \subseteq Q_0 \cup Q_1 \cup \cdots \cup Q_r$$

then $J \subseteq Q_i$ for some i. You can follow the strategy below:

(a) Verify the assertion in the case $r = 0, 1$.

(b) Do induction on r. If $J \subset \bigcup_{j \neq i} Q_j$ then conclude by induction that $J \subset Q_j$ for some j. Otherwise, for each $i = 0, \cdots, n$ choose an element $z_i \in J \setminus \bigcup_{j \neq i} Q_j$. Show that the element $z = z_0 z_1 \cdots z_{r-1} + z_r$ belongs to J but not to any of Q_i.

(6) Prove Lemma 1.5 using Proposition 1.4.