1. The Riemann-Roch Theorem

Last time we discussed:

Proposition 1.1. *Let D, D' be divisors on C. We have*

(1) *If $D \leq D'$ then $L(D) \subseteq L(D')$ and*

$$\dim_\kappa \frac{L(D')}{L(D)} \leq \deg(D' - D).$$

(2) *$L(0) = \kappa$ and $L(D) = 0$ if $\deg(D) < 0$.*

(3) *If $\deg(D) \geq 0$ then $L(D)$ is finite dimensional and $l(D) \leq \deg(D) + 1$.*

(4) *If $D \equiv D'$ then $l(D) = l(D')$.*

On the homework: You studied $l(D)$ when $D = r\,\mathrm{Div}_0(f)$ in the following cases:

(1) $C = \mathbb{P}^1 = \{[X : Y]\}$, $f = Y/X$. You showed

$$\mathrm{Div}_0(f) = [1 : 0], \quad l(r\,\mathrm{Div}_0(f)) = r + 1.$$

Note that $\deg(r\,\mathrm{Div}_0(f)) = r$.

(2) $C = E = V(Y^2Z - X(X - Z)(X - \lambda Z)) \subset \mathbb{P}^2$ is an elliptic curve, $f = Z/X$. You showed

$$\mathrm{Div}_0(f) = 2[0 : 1 : 0], \quad l(r\,\mathrm{Div}_0(f)) = 2r.$$

Note that $\deg(r\,\mathrm{Div}_0(f)) = 2r$.

Today, our goal is to study further the invariant $l(D)$.

Theorem 1.2 (Riemann)**.** *There exists a constant g such that for all divisors D on C we have*

$$l(D) \geq \deg(D) + 1 - g. \tag{1.1}$$

The smallest such g is called the **genus** *of the curve C and is a non-negative integer.*

It can be shown (for instance using sheaf cohomology) that the genus of a curve C is encoded by its Hilbert polynomial $\mathrm{HP}_C(z)$ via the formula

$$g = 1 - \mathrm{HP}_C(0),$$

which shows that the genus of a non-singular plane curve of degree d is given by

$$1 + \frac{d(d-3)}{2} = \frac{(d-1)(d-2)}{2},$$

and in particular a plane cubic has genus 1.

Before proving Theorem 1.2 we need to establish the following technical result:

Lemma 1.3. *Let $f \in K(C) \setminus \kappa$ denote a non-constant rational function, and let D be any divisor on C. For $r \gg 0$ the divisor $r \cdot \mathrm{Div}_0(f) - D$ is linearly equivalent to an effective divisor.*

Proof. Let $\mathcal{Z}(f) \subset C$ denote the set of zeroes of f, and write $D = \sum_P a_P \cdot P$. We consider the rational function $\tilde{f}$ defined by

$$\tilde{f} = \prod_{a_P > 0,\ P \notin \mathcal{Z}(f)} \left(\frac{1}{f} - \frac{1}{f(P)} \right)^{a_P}.$$

By definition, the poles of $\tilde{f}$ are precisely the zeroes of f, and for every $P \notin \mathcal{Z}(f)$ for which $a_P > 0$ we have $\mathrm{ord}_P(\tilde{f}) \geq a_P$. We thus get

$$\mathrm{Div}(\tilde{f}) - D = \mathrm{Div}_0(\tilde{f}) - \mathrm{Div}_\infty(\tilde{f}) - D = \sum_P b_P \cdot P,$$

where a negative coefficient $b_P < 0$ can only occur when $P \in \mathcal{Z}(f)$. This means that

$$r \cdot \mathrm{Div}_0(f) + \sum_P b_P \cdot P \geq 0 \text{ for } r \gg 0,$$

and therefore

$$r \cdot \mathrm{Div}_0(f) - D \equiv r \cdot \mathrm{Div}_0(f) + \mathrm{Div}(\tilde{f}) - D = r \cdot \mathrm{Div}_0(f) + \sum_P b_P \cdot P \text{ is effective when } r \gg 0. \qquad \square$$

Proof of Theorem 1.2. Since $l(0) = 1$ and $\deg(D) = 0$, we conclude that $g \geq 0$ if it exists. We define

$$s(D) = \deg(D) + 1 - l(D)$$

and note that the existence of g is equivalent to the fact that $s(D)$ is bounded above, in which case g is the maximum value of $s(D)$ as D varies among divisors on C.

Note that we have $s(D) = s(D')$ when $D \equiv D'$ by Proposition 1.1(4), and it follows from yesterday's Exercises that $s(D) \leq s(D')$ if $D \leq D'$. In light of Lemma 1.3, it is enough to show that $s(r \cdot \mathrm{Div}_0(f))$ is bounded above when f is some non-constant rational function: indeed, consider a nonconstant rational function $f \in K(C)$ and a positive integer r for which $r \cdot \mathrm{Div}_0(f) - D \equiv E$ for some effective divisor E; it follows that

$$D \equiv r \cdot \mathrm{Div}_0(f) - E \leq r \cdot \mathrm{Div}_0(f)$$

and therefore

$$s(D) = s(r \cdot \mathrm{Div}_0(f) - E) \leq s(r \cdot \mathrm{Div}_0(f)).$$

The fact that $s(r \cdot \mathrm{Div}_0(f))$ is bounded above follows from Proposition 1.6 in yesterday's notes: we have $s(r \cdot \mathrm{Div}_0(f)) \leq \tau + 1$ for all r, concluding the proof. $\square$

Corollary 1.4. *If C is a curve of genus g and if D_0 is a divisor satisfying $l(D_0) = \deg(D_0) + 1 - g$, then for every divisor D which is linearly equivalent to some $D' \geq D_0$ we have*

$$l(D) = \deg(D) + 1 - g.$$

Proof. Using Proposition 1.1(4) we get that $l(D) = l(D')$, so we may assume $D \geq D_0$. It follows that

$$l(D) \leq l(D_0) + \deg(D - D_0) = \deg(D) + 1 - g.$$

If the inequality is strict then we get a contradiction with (1.1), so we must have $l(D) = \deg(D) + 1 - g$. ☐

Corollary 1.5 (Asymptotic Riemann-Roch)**.** *There exists a positive integer N such that for all divisors D with $\deg(D) > N$ we have*

$$l(D) = \deg(D) + 1 - g.$$

Proof. Choose any D_0 such that $l(D_0) = \deg(D_0) + 1 - g$, and let $N = g + \deg(D_0)$. If D is a divisor with $\deg(D) \geq N$ then

$$l(D - D_0) \overset{(1.1)}{\geq} \deg(D - D_0) + 1 - g \geq N - \deg(D_0) + 1 - g = 1,$$

so that there exists $0 \neq f \in L(D - D_0)$. By definition, this means that

$$D - D_0 + \mathrm{Div}(f) \geq 0, \text{ i.e. } D + \mathrm{Div}(f) \geq D_0.$$

Set $D' = D + \mathrm{Div}(f)$ and note that $D \equiv D'$, hence by Corollary 1.4 we get $l(D) = \deg(D) + 1 - g$, as desired. ☐

Theorem 1.6 (Riemann-Roch)**.** *Let C be a non-singular projective curve. There exists a divisor W (which is called a* **canonical divisor***) with the property that for any divisor D on C we have*

$$l(D) - l(W - D) = \deg(D) + 1 - g.$$

Corollary 1.7. *If W is a canonical divisor on a curve C of genus g then*

$$\deg(W) = 2g - 2 \textit{ and } l(W) = g.$$

As a consequence, we can take $N = 2g - 2$ in Corollary 1.5.

Proof. Replacing D by $W - D$ in the Riemann-Roch formula yields

$$l(W - D) - l(D) = \deg(W - D) + 1 - g$$

and therefore

$$\deg(D) + 1 - g = -\deg(W - D) - 1 + g = \deg(D) - \deg(W) - 1 + g$$

which proves $\deg(W) = 2g - 2$.

We can now set $D = W$ in the Riemann-Roch formula and use the fact that $l(0) = 1$ to conclude that

$$l(W) = l(0) + \deg(W) + 1 - g = 1 + 2g - 2 + 1 - g = g.$$

To prove the last statement, consider a divisor D with $\deg(D) > 2g - 2$. We get that $\deg(W - D) < 0$ which by Proposition 1.1(2) yields $l(W - D) = 0$. The Riemann-Roch formula gives then the conclusion of Corollary 1.5. □

When $C = E$ is an elliptic curve we can in fact take $W = 0$ (see Exercise 2), but Corollary 1.7 only guarantees that $\deg(W) = 0$, since $g = 1$. Nevertheless, this suffices to give a proof of the following.

Theorem 1.8. *Suppose that $E \subseteq \mathbb{P}^2$ is a non-singular cubic curve and fix a point $O \in E$. We have a bijection*

$$\phi : E \longrightarrow \mathrm{Cl}^0(E), \quad \phi(P) = P - O.$$

Proof. We first show that the map ϕ is surjective. Consider a divisor D with $\deg(D) = 0$ and let $D' = D + O$. Since $\deg(W) = 0$ and $\deg(D') = 1$ it follows that $l(W - D') = 0$ and therefore by Riemann-Roch

$$l(D') = 1.$$

It follows that there exists an effective divisor D'' linearly equivalent to D', and since $\deg(D'') = 1$ it must be that $D'' = P$ for some point $P \in E$. This shows that $D + O \equiv P$, or equivalently that $D \equiv P - O = \phi(P)$ proving surjectivity.

To see that ϕ is injective, assume that $\phi(P) = \phi(Q)$ for distinct points $P, Q \in E$. This means that $P - Q \equiv 0$ and therefore $l(P - Q) = 1$. We can thus find a rational function $f \in K(E)$ with $\mathrm{Div}(f) = Q - P$, which in particular is non-constant. This implies that $1, f \in L(P)$ are linearly independent over κ, thus $l(P) \geq 2$. However, Riemann-Roch implies that $l(P) = 1$, which is a contradiction. □

1.1. **Exercises.**

(1) Let $C = \mathbb{P}^1$ (genus $g = 0$). Explain why you can take the canonical divisor W to be any divisor of degree -2 in the Riemann–Roch theorem.

(2) (a) Show that in the Riemann–Roch theorem, you can replace W with any linearly equivalent divisor $W' \equiv W$.

(b) Suppose that E is an elliptic curve (genus $g = 1$). Use the fact that $l(W) = g$ to show that W is linearly equivalent to an effective divisor, and determine this divisor using $\deg(W) = 2g - 2$.

(c) Conclude that for an elliptic curve, you can take $W = 0$ in the Riemann–Roch theorem.

(3) Let E be an elliptic curve, and fix $O \in E$. The goal of this exercise is to show (see also the notes) that we have a bijection

$$\phi : E \longrightarrow \mathrm{Cl}^0(E), \quad \phi(P) = P - O.$$

- Surjectivity: every element of $\mathrm{Cl}^0(E)$ is the equivalence class of a divisor D with $\deg(D) = 0$. Use Riemann–Roch to conclude that $l(D + O) = 1$, and conclude that $D + O$ is equivalent to an effective divisor of degree 1 (what does such a divisor look like?).
- Injectivity: show that if $P - O \equiv Q - O$, $P \neq Q$, then $L(P - Q)$ contains a non-constant rational function f. Explain why this implies $1, f \in L(P)$, and $l(P) \geq 2$. Show that this contradicts Riemann–Roch, so ϕ is in fact injective.

(4) Let C be a curve of genus g and let $P \in C$. Show that for every $a \geq 2g$ there exists a rational function $f \in K(C)$ with

$$\mathrm{Div}_\infty(f) = a \cdot P.$$

Prove that the above conclusion fails when a is small (for instance when $a = 1$ and C is any curve of genus $g \geq 1$).

(5) Let C be a curve of genus g, let $P \in C$, and define

$$N_r = l(rP) \text{ for } r \geq 0.$$

(a) Show that $1 = N_0 \leq N_1 \leq \cdots \leq N_{2g-1} = g$ and conclude that there are precisely g numbers

$$0 < a_1 < a_2 < \cdots < a_g < 2g$$

with the property that there is no rational function $f \in K(C)$ with $\mathrm{Div}_\infty(f) = a_i \cdot P$.

The numbers $a_1, \cdots, a_g$ are called **Weierstrass gaps**, and $(a_1, \cdots, a_g)$ is the **gap sequence at** P. We say that P is a **Weierstrass point** if the gap sequence is different from $(1, 2, \cdots, g)$.

(b) Show that the following are equivalent

- P is a Weierstrass point.
- $l(gP) > 1$.
- $l(W - gP) > 0$.

(c) Show that if a and b are not gaps then $a + b$ is not a gap. Conclude that if 2 is not a gap then the gap sequence is necessarily $(1, 3, \cdots, 2g-1)$. The curve C is called **hyperelliptic** in this case.