Minicourse on Modular Forms

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Lecture 1
Monday

Motivation The Dedekind delta function is the formal power series in the variable q:

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - \cdots = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Ramanujan observed (dreamed?) that

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

Out story begins, like many things in modern number theory, with the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Some special values $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, and in general $\zeta(2k)$ is a rational multiple of π^{2k} (see exercise set). I say that our story begins here because 691 is, in fact, the numerator of the rational number in $\zeta(12)$. In general, these rational scalars contain a lot of number theoretic properties, but they are just constants.

The connection between Δ and divisor sums, it turns out, goes via complex analysis by reinterpreting the formal variable q as a complex function $q = e^{2\pi i z}$, and extending the sum defining $\zeta(s)$ to a sum that also contains z.

1 Special complex functions

Suppose $z \in \mathbb{C}$ and $k \geq 2$ is an integer. Define

$$G_{2k}(z) = \sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^{2k}}.$$

Question 1. *What kinds of z can we plug in? And what kinds of k? And what does it mean to have a converging double series?*

The sum converges absolutely when $2k \geq 4$ and $\operatorname{Im} z > 0$ and can be expressed as a (Fourier) power series in $q = e^{2\pi i z}$.

Proposition 2.

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(-2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n.$$

This expression is very transcendental, but if we rescale we get

$$E_{2k} = \frac{(2k-1)!}{2(-2\pi i)^{2k}} G_{2k} = \frac{(2k-1)!\zeta(2k)}{(-1)^k (2\pi)^{2k}} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n.$$

Amazingly, these power series (called **Eisenstein series**) in q have rational coefficients

$$E_{2k} \in \mathbb{Q}[\![q]\!],$$

as $\zeta(2k)$ is a rational multiple of π^{2k} (see today's exercise set).

Application

How will this help us at all to show Ramanujan's congruence? Looking at the first few Eisenstein series

$$\begin{aligned} E_4 &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \cdots \\ E_6 &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + 3126q^5 + \cdots \\ E_{12} &= \frac{691}{65520} + q + 2049q^2 + 177148q^3 + 4196353q^4 + 48828126q^5 + \cdots \end{aligned}$$

we'll be able to show the following stunning identity:

$$\Delta = E_{12} - \frac{691}{13} \left(1600 E_4^3 + 21 E_6^2 \right) \tag{1.1}$$

of rational power series in q. Comparing the coefficients of q^n mod 691 is precisely the Ramanujan identity.

How on earth will we prove identity (1.1)? It turns out that Δ, E_{12}, E_4^3, and E_6^2 are special functions called **modular forms**, which are part of a vector space of dimension 2.

Proof of Proposition 2

Lemma 3 (See exercise set)**.** *If* $z \in \mathbb{C}$ *then*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(2k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}.$$

We'll separate the double sum defining G_{2k} into a part where $m = 0$ and a part where $m \neq 1$:

$$\begin{aligned}
G_{2k} &= \sum_{n\neq 0} \frac{1}{n^{2k}} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^{2k}} \\
&= 2\zeta(2k) + 2\sum_{m=1}^{\infty} \frac{(-2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} n^{2k-1} e^{2\pi i n m z} \\
&= 2\zeta(2k) + 2\frac{(-2\pi i)^{2k}}{(2k-1)!} \sum_{m,n\geq 1} n^{2k-1} q^{mn} \\
&= 2\zeta(2k) + 2\frac{(-2\pi i)^{2k}}{(2k-1)!} \sum_{N=1}^{\infty} \left(\sum_{n|N} n^{2k-1} \right) q^N.
\end{aligned}$$

2 Modular Forms

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. This is the natural domain for the functions G_{2k}.

Definition 4. A modular form of weight $2k$ is a function $f : \mathcal{H} \to \mathbb{C}$ which satisfies the following important conditions:

1. $f(z)$ is analytic ("power series in z"),
2. $f(z)$ can be expressed as a power series in $q = e^{2\pi i z}$, and
3. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z)$ for all $z \in \mathcal{H}$ and all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

On the exercise set we'll see that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We'll see an exercise that if the functional equation is satisfied for $g_1, g_2 \in \mathrm{SL}_2(\mathbb{Z})$ then the same is true for $g_1 g_2$. This means that to check the functional equation condition, we only need to check it for T and S. The condition is automatic for T because $q(z+1) = q(z)$ and f is a power series in q, and for S we need to check that

$$f(-\frac{1}{z}) = z^{2k} f(z).$$

Therefore, the last condition can be replaced with

$$f(-\frac{1}{z}) = z^{2k} f(z).$$