Lecture 2
Tuesday

We denote M_{2k} the complex vector space of modular forms of weight $2k$.

Example 5. The function $G_{2k}(z)$ is a modular forms of weight $2k$.

Aside on analytic functions

Why are power series so important? And what do we mean by "power series"? Let's remind ourselves the statement of Green's theorem:

$$\oint_C (Pdx + Qdy) = \iint_D (Q_x - P_y)dxdy,$$

for any simple closed curve C oriented to have its interior D on the left, and any P and Q with partial derivatives defined in D.

Suppose now we have a function $f(z)$ of a complex variable z. We'd like to be able to define derivatives of $f(z)$ and to compute integrals of the form $\int f(z)dz$, by which we mean integrals of the form $\int f(x+iy)(dx+idy)$. Let's try to do this for $f(z) = z^n$ over an oriented simple closed curve C. Then $\int_C z^n dz = \oint_C (x+iy)^n(dx+idy)$, which is precisely of the kind that shows up in Green's theorem with $P(x,y) = z^n$ and $Q(x,y) = iz^n$:

$$\begin{aligned}\int_C z^n dz &= \oint_C z^n dx + iz^n dy \\ &= \iint_D (Q_x - P_y)dxdy.\end{aligned}$$

Compute Q_x and P_y, and you see that you get

$$\begin{aligned}P_y &= \frac{\partial z^n}{\partial y} = inz^{n-1} \\ Q_x &= i\frac{\partial z^n}{\partial x} = inz^{n-1}\end{aligned}$$

are equal, which means that the integral above vanishes!

By adding monomials we also get that if $f(z) = \sum a_n z^n$ then $\int f(z)dz = 0$ over any oriented simple closed curve because

$$f_y = if_x.$$

In complex analysis this relation, which remember captures that we integrate the 0 function in Green's theorem, has a fancy name: the **Cauchy-Riemann equation**. It's a fundamental result in complex analysis that if $f(z)$ is a function defined on an open set $U \subset \mathbb{C}$ with no holes and $f(z)$ satisfies this equation (in other words integrating over oriented simple closed curves we get 0) then

1. $f'(z) = f_x(z) = i^{-1}f_y(z)$ makes sense, and

2. for any $u \in U$, $f(z)$ is a power series in $z - u$ around u.

Back to Example 5

Why is G_{2k} a modular form of weight $2k$?

- Analyticity on $\mathcal{H}$: Check the partials! (And convergence.)
- Analyticity at $i\infty$ comes for free from Proposition 2.
- Functional equations.

An example which is not Eisenstein

Example 6. The Delta function $\Delta(z)$ is a modular form of weight 12.

We learn in calculus that log differentiation is a powerful way of dealing with products. Let's do that

$$\begin{aligned}\frac{d\log\Delta(z)}{dz} &= \frac{d\Delta(z)}{\Delta(z)}\\ &= \frac{dq}{dz} + 24\sum_{n=1}^{\infty}\frac{d\log(1-q^n)}{dz}\\ &= 2\pi iz - 24\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{dq^{mn}}{mdz}\\ &= 2\pi iz - 24\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{mn}\\ &= 2\pi iz - 24\sum_{n=1}^{\infty}\left(\sum_{d|n}d\right)q^n.\end{aligned}$$

This formula is reminiscent of the non-constant part of G_{2k} but if $2k$ were 2. We only made sense of G_{2k} when $2k \geq 4$ because of absolute convergence, but it turns out one can define

$$G_2(z) = \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}{}'\frac{1}{(mz+n)^2}\right),$$

and this converges whem $\operatorname{Im} z > 0$. Here $\sum'$ means that m and n cannot both be 0. Because the series converges, but not absolutely, we cannot change the order of summation!

In fact, the proof of Proposition 2 still works and we get

$$G_2 = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)q^m,$$

so $\frac{d\log\Delta(z)}{dz} = \frac{6i}{\pi}G_2(z)$.

The function $G_2(z)$ is NOT modular (the order of summation matters!) but check that $G_2(-\frac{1}{z}) = z^2 G_2(z) - 2\pi i z$. (See exercise set.) This gives

$$\frac{d \log \Delta(-\frac{1}{z})}{dz} = \frac{6i}{\pi}\left(z^2 G_2(z) - 2\pi i z\right) \cdot \frac{1}{z^2} = \frac{d \log \Delta(z)}{dz} + \frac{12}{z},$$

which gives $\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$ up to a constant. How do you find the constant? Plug in $z = i$!