3 The vector space of modular forms

The set of modular forms M_{2k} is a complex vector space, which a priori tells us nothing. To get arithmetic what we really need is that M_{2k} is **finite dimensional**.

We will use an argument of Zagier's to show the following:

Theorem 7. $\dim M_{2k} \le \frac{2k}{12} + 1.$

The proof will require us to think imaginatively about the upper half plane.

Cut and Paste

In the real setting, the exponential map $e^x : \mathbb{R} \to (0, \infty)$ is one-to-one and onto and its inverse is $\log x : (0, \infty) \to \mathbb{R}$.

In the complex setting, we can still write $e^z : \mathbb{C} \to \mathbb{C}^\times$. Is it one-to-one? Is it onto? How about $\log z$ for $z \in \mathbb{C}^\times$?

[Demonstration]

What about $\mathcal{H}$? Can we make sense of $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$?

Lemma 8. *A fundamental domain for $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$ is the region $\mathcal{F}$ consisting of $z \in \mathcal{H}$ with $-\frac{1}{2} < \operatorname{Re} z \le \frac{1}{2}$ and $|z| \ge 1$ when $\operatorname{Re} z \ge 0$ and $|z| > 1$ when $\operatorname{Re} z < 0$.*

This means that any modular form $f(z) \in M_{2k}$ is completely characterized by its values on $\mathcal{F}$.

We can glue $\mathcal{F}$ to produce a punctured sphere, which is the associated Riemann surface to $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$. There's a catch however: to make this identification geometrically meaningful, for instance to make sure that distances/areas measured in $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$ be the same as distances/areas measured on the sphere, we need to stretch $\mathcal{H}$ to make it analytically the Riemann sphere.

The main computational result is the following:

Proposition 9. *Suppose $f(z)$ is a modular form of weight $2k$, with zeros at $P_1, \ldots, P_d$ in the interior of $\mathcal{F}$. Then*

$$\sum \operatorname{ord}_{P_i}(f) \le \frac{2k}{12}.$$

The idea is that any analytic function integrates to 0 on any simple closed curve.

Integrating functions with poles

What about functions $f(z)$ which are not power series? We can deal with functions of the form

$$f(z) = \frac{c}{z-u} + a_0 + a_1(z-u) + a_2(z-u)^2 + \cdots,$$

in which case the function $f(z)$ is said to have a simple pole at u with **residue** $\operatorname{res}_u f(z) = c$.

Lemma 10 (Cauchy integral)**.** *Say C is an simple closed curve around u, oriented counter-clockwise, and $f(z)$ has a simple pole at u. Then*

$$\int_C f(z)dz = 2\pi i \operatorname{res}_u f(z).$$

Proof of Proposition 9

Allow $f(z)$ to possibly have poles on the vertical side of $\mathcal{F}$, at $i\infty$, at i, at ζ_3, and on the bottom arc of $\mathcal{F}$. [Demonstration of the integral]

Proof of Theorem 7

Suppose $d = \lceil \frac{2k}{12} \rceil$. Let's assume, by contradiction, that $\dim M_{2k} > d$. This means we can find $d+1$ modular forms which are independent over $\mathbb{C}$, say they are $f_1, f_2, \ldots, f_{d+1}$.

Pick d random points inside $\mathcal{F}$ (not on the boundary), $P_1, \ldots, P_d$, and look at the evaluations

$$\begin{pmatrix} f_1(P_1) & f_2(P_1) & \ldots & f_{d+1}(P_1) \\ f_1(P_2) & f_2(P_2) & \ldots & f_{d+1}(P_2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(P_d) & f_2(P_d) & \ldots & f_{d+1}(P_d) \end{pmatrix}$$

By linear algebra, we can always find a linear combination of the $d+1$ colums of this $d\times(d+1)$ matrix which is 0. This means we can find some scalars $a_1, \ldots, a_{d+1}$ (not all 0) such that

$$a_1 \,\text{column}_1 + a_2 \,\text{column}_2 + \cdots + a_{d+1} \,\text{column}_{d+1} = 0$$

which means $f(z) = a_1 f_1(z) + \cdots + a_{d+1} f_{d+1}(z)$ has zeros at $P_1, \ldots, P_d$. But this is impossible by Proposition 9!

Application

We conclude that

$$\dim M_{12} \le \frac{12}{12} + 1 = 2.$$

Since $E_4^3, E_6^2 \in M_{12}$ are linearly independent (check!), it follows that $\Delta - E_{12} = aE_4^3 + bE_6^2$ for some scalars a and b. We can compute a, b by looking at the first few coefficients of the q-power series. (See exercise set.)