Another approach to counting dimension

This is short. Here's another approach to counting dimensions of modular forms, that works in great generality.

Pick $f_0 \in M_{2k}$. If $f \in M_{2k}$ is any other modular form, then you can look at the quotient $g = f/f_0$. This function is NOT analytic, only meromorphic, but it satifies that $g(\gamma \cdot z) = g(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This means that g is, in fact, a function on the punctured Riemann sphere $X = \mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$, as well as at the point at infinity.

Suppose you chose a random meromorphic function g. You could always compute $f = gf_0$. This function satisfies the functional equations, but is it a modular form? To be so, you'd need gf_0 to be analytic so you'd need $\operatorname{div}(gf_0)$ to be effective, in other words, that $\operatorname{div} g + \operatorname{div} f_0 \geq 0$. We conclude that M_{2k} is in bijection with $\mathcal{L}(\operatorname{div} f_0)$. The dimension of this space was the topic of today's algebraic curves lecture, and Riemann-Roch is a great way to compute this dimension exactly, as opposed to finding an upper bound only, as we did in the "integrals on the balloon computation".

Modular forms of level N

We've been dealing with modular forms in M_{2k} and we showed a very strong result about how to obtain all modular forms: $\dim M_{2k}$ is roughly $\frac{2k}{12}$ and the modular forms in M_{2k} are all of the form $E_{2k-12\ell} \cdot \Delta^{\ell}$. This is enough to prove some beautiful relations, but it is not the whole story.

Definition 11. Suppose $N \geq 1$. A modular form of weight $2k$ and level N is a function $f : \mathcal{H} \to \mathbb{C}$ with the same properties as M_{2k} but we only impose the functional equation

$$f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $N \mid c$ (these matrices form a subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$). We write $M_{2k}(N)$ for the space of such modular forms.

Theta series

Writing down modular forms is hard because these functions have to be power series in q that satisfy very strong functional equations. We've seen one example, the Eisenstein series, that we wrote down as a converging series in z and then, miraculously, showed to be arithmetically meaningful q-power series.

But showing that, for instance, Δ was a modular form was hard, and we could only do it because Δ was a predictable product of q-s, and not a predictable q-power series.

There is, however, one source of predictable power series in q that has nice functional equation properties:

Definition 12. The **theta series** is

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}.$$

When does this converge? (Incidentally, this theta series is useful in showing that the Riemann zeta function $\zeta(s)$ can be defined for all complex numbers s except 1.)

Lemma 13. *For any z: $\theta(-\frac{1}{4z}) = \sqrt{\frac{2z}{i}}\theta(z)$. (This is a standard computation in Fourier transforms, one of the simplest applications of the Poisson summation formula.)*

Clearly θ is NOT a modular form. However:

Proposition 14. *$\theta^4 \in M_2(4)$ is a modular form of weight 2 and level 4.*

Proof. First, θ is a power series in q and it converges everywhere, so it has all the analytic properties needed. The only thing to check are the functional equations.

For level 1 modular forms, the infinitely functional equations could be disposed of by checking only two: for T and S, namely that $f(z+1) = f(z)$ and $f(-\frac{1}{z}) = z^{2k} f(z)$. We checked this because the matrices T and S generated all of $\mathrm{SL}_2(\mathbb{Z})$. In general, it is still true that $\Gamma_0(N)$ is generated by finitely many matrices, but typically many more than 2. However, it's still the case that $\Gamma_0(4)$ is generated by T and $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.

The functional equation for T is just that θ (and therefore θ^4) is a power series in q. What about the functional equation for A? We'd have to check that

$$\theta^4(\frac{1}{4z+1}) = (4z+1)^2 \theta^4(z),$$

which looks so different from what we know about θ that it seems we won't be able to get anywhere.

Instead, time for a little playing around, to notice that

$$A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}^{-1}. \tag{3.1}$$

We already know from the exercise set that if the functional equation is true for some matrices, it is true for their product (and inverse). So it would be enough to check the functional equation for the matrix $\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$, which is profoundly weird because it doesn't have integral coefficients! Nevertheless, the functional equation still makes sense and in fact, the functional equation is precisely

$$\theta^4(\frac{-\frac{1}{2}}{2z}) = \theta^4(-\frac{1}{4z}) = \left(\sqrt{\frac{2z}{i}}\right)^4 \theta^4(z) = -(2z)^2\theta^4(z).$$

Unfortunately, we are off by a sign. That's fine, however, because in (3.1), we see this weird matrix twice, so the two negative signs cancel out! □

Application: sums of four squares

This took long, but we can now apply the fact that $\theta^4 \in M_2(4)$, a vector space of dimension 2 (similar to what we've done for level 1), and try to write θ^4 explicitly in terms of Eisenstein series, similar to what we did in the case of Δ.

What Eisenstein series? In the case of level 1, we saw that G_2 converged only conditionally, and was definitely not a modular form (it didn't satisfy the functional equation for S!).

We use a trick that turns out to be extremely important in number theory:

$$G_{2,2}(z) = G_2(z) - 2G_2(2z)$$
$$G_{2,4}(z) = G_2(z) - 4G_2(4z)$$

On the exercise set you will check that $G_{2,2}$ and $G_{2,4}$ are in $M_2(4)$.

Rescaling by $8\pi^2$ we get

$$\begin{aligned}
E_2 &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n \\
E_{2,2} &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n - \left(-\frac{2}{24} + \sum_{n=1}^{\infty} 2\sigma_1(n)q^{2n}\right) \\
&= \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n - \sum_{n=1}^{\infty} 2\sigma_1(n)q^{2n} = \frac{1}{24} + q + q^2 + \cdots \\
E_{2,4} &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n - \left(-\frac{4}{24} + \sum_{n=1}^{\infty} 4\sigma_1(n)q^{4n}\right) \\
&= \frac{3}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n - \sum_{n=1}^{\infty} 4\sigma_1(n)q^{4n} = \frac{3}{24} + q + 3q^2 + \cdots .
\end{aligned}$$

Visibly, $E_{2,2}$ and $E_{2,4}$ are not linearly dependent, so they must be a basis of $M_2(4)$. We deduce

$$\theta^4 = aE_{2,2} + bE_{2,4},$$

for some scalars a and b. Playing around with a few coefficients we get

$$\theta^4 = 1 + 8q + 24q^2 + \cdots = 8E_{2,4}.$$

At the same time

$$\theta^4 = \sum_N r(N)q^N = 1 + \sum_n \sigma_1(n)q^n - \sum_n 4\sigma_1(n)q^{4n}$$

so the number $r(N)$ of ways of writing $N = x^2 + y^2 + z^2 + t^2$ is

$$r(N) = \sum_{d|N, 4\nmid d} d.$$

Tying back to elliptic curves

On Wednesday's exercise set, you looked at the elliptic curve

$$y^2 + y = x^3 - x.$$

What are a_2, a_3, a_5? What is the L-function of this elliptic curve? [Show on lmfdb]

A most amazing connection is satisfied between elliptic curves and modular forms:

Theorem 15 (Modularity theorem)**.** *If $L(E, s) = \sum a_n n^{-s}$ is the L-function of an elliptic curve then $f = \sum a_n q^n$ is a modular forms of weight 2 and some level (which depends on the elliptic curve).*