$|X| = m \quad |Y| = n$

Pset 1

1). Show By induction $\#\{f : X \to Y\}$ is $n(n-1)\cdots(n-m+1)$
f bijective

(a)

so $\# S_n = n!$

Pick $x \in X$

$$\#\{f : X \to Y \text{ bij}\} = \sum_{y \in Y} \#\{f : X \to Y \text{ bij } f(x) = y\}$$

$$= \sum_{y \in Y} \#\{f : X - \{x\} \to Y - \{y\} \text{ bij } f(x) = y\}$$

$$\overset{\text{by induction}}{=} \sum_{y} (n-1)\cdots(n-1-(m-1)+1)$$

$$= n(n-1)\cdots(n-m+1)$$

Base case is $\# X = m = 1$ in which case have n choices.

(b) $S_3 \subset S_n$ S_3 is not abelian as $(12)(23) \neq (23)(12)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

so S_n not abelian.

2). If $a \in H$ then $n = \text{ord}(a) < \infty$ as H is finite

so $\langle a \rangle = \{1, a, \ldots, a^{n-1}\}$ with $a^{n-1} = a^{-1}$

so $a^{-1} \in H$ too. $\forall a, b \in H$, $b^{-1} \in H$

so $a \cdot b^{-1} \in H$ so H subgroup.

3). (b) Symmetries of n-gon are bijections of the n-vertices

so $D_{2n} \subset S_n$ as sets. The binary operations

are composition of bijections in both cases so

$D_{2n} < S_n$ is a subgroup. it is proper as $2n = |D_{2n}| < n! = |S_n|$

(a) $D_6 < S_3$ by (b) and both have 6 elements so $D_6 = S_3$

4). a) $\Rightarrow$ $T_{g,h} = g \cdot h$ gives the table $(T_{g,h})$

then $T_{e,h} = h$ $T_{g,e} = g$ so

T has row & column with required properties

Finally, row g has entries $\{T_{g,h} = g \cdot h\}$ which are all distinct as $gh = gh' \Rightarrow h = h'$. Similarly for columns.

$\Leftarrow$ Suppose $T_{g,h}$ satisfies these properties, for the $g \cdot h := T_{g,h}$ defines a semigroup G. To check group law on G check $\exists e$ and $\exists g^{-1}$.

For e take the row label e in which case

$$eg = ge = g.$$

To show g^{-1} exists:

Pick g. row g contains $\{g \cdot h \mid h \in G\}$ and each element of G appears exactly once. Thus e appears so $\exists$ $h = g^{-1}$ st $gh = e$ so $gg^{-1} = e$.

Need $g^{-1}g = e$ too. But $g^{-1}gg^{-1} = g^{-1}e = g^{-1}$

so look at the column $\{T_{h,g^{-1}}\} = \{hg^{-1}\}$. and $g^{-1}gg^{-1} = g^{-1} = eg$ as g^{-1} appears exactly once

deduce $g^{-1}g = e$.

(b) $|G| = 1$ $T = \{e\}$

$|G| = 2$ $T =$

	e	g
e	e	g
g	g	e

$|G| = 3$ $G = \{e, a, b\}$

	e	a	b
e	e	a	b
a	a	b	$\boxed{e}$
b	b	$\boxed{e}$	a

not b or a so e

first $\square$ then $\bigcirc$

$|G| = 4$ $\qquad G = \{e, a, b, c\}$

first ○

two cases

	e	a	b	c
e	e	a	b	c
a	a		○	
b	b			
c	c			

e or c

Case 1

first □
second ○

	e	a	b	c
e	e	a	b	c
a	a	c 2	e	$\boxed{b}$ 1
b	b	e 3	c 6	a 7
c	c	b 4	a 5	e 8

$e \mapsto 0$
$a \mapsto 1$
$c \mapsto 2$
$b \mapsto 3$

gives $\mathbb{Z}/4\mathbb{Z}$

Case 2

	e	a	b	c
e	e	a	b	c
a	a		c	□
b	b			
c	c			

e or b

Case 2.1

	e	a	b	c
e	e	a	b	c
a	a	c 2	c	b 1
b	b	c 3		
c	c	b 4		

after step4 $b^2 = bac = c^2$
associative

$cb = cac = bc$

so

	b	c
b		
c		

either $\begin{matrix} a & e \\ e & a \end{matrix}$

or $\begin{matrix} e & a \\ a & e \end{matrix}$

Case 2.2

	e	a	b	c
e	e	a	b	c
a	a	b 2	c	e 1
b	b	c 3	e 6	a 5
c	c	e 4	a 7	b 8

after Step 2 $ba = a^3$
$= ab = c$

$e \mapsto 0$
$a \mapsto 1$
$b \mapsto 2$
$c \mapsto 3$

$\mathbb{Z}/4\mathbb{Z}$

2.1.1 Case $\begin{matrix} e & a \\ a & e \end{matrix}$

$e \mapsto (0,0)$
$a \mapsto (1,0)$
$b \mapsto (0,1)$
$c \mapsto (1,1)$

gives $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

2.1.2 Case $\begin{matrix} a & e \\ e & a \end{matrix}$

$e \mapsto 0$
$a \mapsto 2$
$b \mapsto 1$
$c \mapsto 3$

gives $\mathbb{Z}/4\mathbb{Z}$