Graduate Algebra
Homework 2

Fall 2014

Due 2014-09-10 at the beginning of class

1. Show that the dihedral group D_8 with 8 elements is isomorphic to the subgroup of $\mathrm{GL}(2,\mathbb{R})$ generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2. Let Q be the subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

   (a) Show that Q is a non-abelian group with 8 elements.

   (b) Show that Q and D_8 are not isomorphic.

   (c) Show that all the subgroups of Q are normal.

   Q is known as the **quaternion group**.

3. (a) Show that every subgroup of $\mathbb{Z}$ is infinite cyclic.

   (b) Show that every finite subgroup of $\mathbb{C}^\times$ is of the form $\mu_n = \{z \in \mathbb{C} | z^n = 1\}$. [Hint: Proof from (a) also works for (b).]

4. Let p be a prime number and G the set of upper triangular 3×3 matrices with 1-s on the diagonal and entries in $\mathbb{Z}/p\mathbb{Z}$.

   (a) Show that G is a group with respect to matrix multiplication, where addition and multiplication in $\mathbb{Z}/p\mathbb{Z}$ are taken modulo p.

   (b) Show that $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$.

   (c) Show that $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$?

   G is known as a **Heisenberg group** (think entry 12 as position and entry 23 as momentum in quantum mechanics) which is an example of an **extraspecial group**. Both D_8 and Q above are extraspecial groups as well.

5. Let $K \subset H$ be two subgroups of a group G. Show that $[G : K] = [G : H][H : K]$.