Graduate Algebra
Homework 2

Fall 2014

Due 2014-09-10 at the beginning of class

1. Show that the dihedral group D_8 with 8 elements is isomorphic to the subgroup of $\mathrm{GL}(2,\mathbb{R})$ generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Proof. First, remark that if G is a group, $a, b \in G$ such that $\{a^k b^l\}$ is a group then $\langle a, b\rangle = \{a^k b^l\}$. Indeed, $\langle a, b\rangle$ is the smallest group containing a, b and certainly contains $\{a^k b^l\}$. Let a be the first matrix and b the second. Then $a^2 = -I_2$, $b^2 = I_2$ and $ab = -ba$. This implies that $a^{-1} = -a$, $b^{-1} = b$ and so $a^k b^l (a^m b^n)^{-1} = a^k b^{l-n} a^{-m}$ is either a^{k-m} or $a^k b a^{-m} = (-1)^m a^{k-m} b = a^{k+m} b$. Thus $\langle a, b\rangle = \{a^k b^l\}$ and this set contains, as $a^4 = 1, b^2 = 1$ only the elements $\{1, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$. The map $a \mapsto R$, $b \mapsto F$ gives an isomorphism with D_8. □

2. Let Q be the subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

   (a) Show that Q is a non-abelian group with 8 elements.

   (b) Show that Q and D_8 are not isomorphic.

   (c) Show that all the subgroups of Q are normal.

   Q is known as the **quaternion group**.

   Proof. (1) As before, a is the first matrix, b is the second. Then $a^2 = b^2 = -1$ and $ab = -ba$ so $\{a^k b^l\}$ is again a group. It is not abelian and as a set it is $\{1, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$.

   (2): Under a group isomorphism the orders of elements are preserved. But R^2 and F have order 2 in D_8 so there must be two elements of order 2 in Q. However, only $a^2 = b^2 = -I_2$ has order 2 in Q so there cannot be any isomorphism.

   (3): Two ways: either list all subgroups and check manually that they are all normal (1, Q are; $\langle a\rangle, \langle b\rangle, \langle ab\rangle$ are normal; $\langle a^2 = b^2\rangle$ is normal) OR show that every index 2 subgroup must be normal. Since $|Q| = 8$ the only remaining case is the unique subgroup $\langle -1\rangle$ of order 2 which is normal because $-1 \in Z(Q)$.

   For the first method you really need to check normality by hand: $gH = Hg$ for each subgroup H and each $g \in Q$. I won't write it out, but it's not difficult. For the second method, suppose $[G : H] = 2$. Then $G = H \sqcup gH$. For each $h \in H$ check $hH = Hh$ and $ghH = Hgh$. The first one is clear. For the second one need $gH = Hg$. Suppose $ag \in Hg$ for $a \in H$ but $ag \notin gH$. Then $ag \in H$ and so $g \in H$ which cannot be. Thus $Hg \subset gH$. The opposite inclusion is similar and H is normal. □

3. (a) Show that every proper subgroup of $\mathbb{Z}$ is infinite cyclic.

   (b) Show that every finite subgroup of $\mathbb{C}^\times$ is of the form $\mu_n = \{z \in \mathbb{C} | z^n = 1\}$. [Hint: Proof from (a) also works for (b).]

Proof. (1): H proper means there is a nonzero element n of lowest absolute value. Since $\pm n \in H$ assume $n > 0$. Let $m \in H$ and write $m = nq + r$ with $0 \le r < n$. Then $m - nq = r \in H$ and by minimality of n, $r = 0$. Thus $H \subset n\mathbb{Z}$. But $n \in H$ and so $H = n\mathbb{Z}$.

(2): Since G is finite, $g^{|G|} = 1$ for all $g \in G$ and so $G \subset \mu_{|G|}$. But $|\mu_{|G|}| = |G|$ and so the inclusion is an equality. □

4. Let p be a prime number and G the set of upper triangular 3×3 matrices with 1-s on the diagonal and entries in $\mathbb{Z}/p\mathbb{Z}$.

   (a) Show that G is a group with respect to matrix multiplication, where addition and multiplication in $\mathbb{Z}/p\mathbb{Z}$ are taken modulo p.

   (b) Show that $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$.

   (c) Show that $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$?

   G is known as a **Heisenberg group** (think entry 12 as position and entry 23 as momentum in quantum mechanics) which is an example of an **extraspecial group**. Both D_8 and Q above are extraspecial groups as well.

   Proof. Write $(a, c; b) = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$.

   (1) Since $G \subset \mathrm{GL}(3, \mathbb{Z}/p\mathbb{Z})$ it suffices to check that $(a, c; b)(x, z; y)^{-1} \in G$. But $(x, z; y)^{-1} = (-x, -z; xz - y)$ and $(a, c; b)(-x, -z; xz - y) = (a - x, c - z; b - cz + xz - y) \in G$.

   (2): Suppose $(a, c; b) \in Z(G)$. Then $(a, c; b)(x, z; y) = (x, z; y)(a, c; b)$ for all x, y, z. Thus $b + az + y = y + zc + b$ for all x, y, z and so $a = c = 0$. Reciprocally, $(0, 0; b) \in Z(G)$ which is easy to see. Thus $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$ sending $(0, 0; b) \mapsto b$, which is clearly an isomorphism.

   (3): Need to find representatives of $G/(G)$ in G, by "making $Z(G)$ vanish". Thus we ignore b and get the map $(a, c; b)Z(G) \mapsto (a, c) \in (\mathbb{Z}/p\mathbb{Z})^2$. This is well defined because $(a, c; b)(0, 0; x) = (a, c; b + x)$ and so a and c don't depend on the element of the coset $(a, c; b)Z(G)$. Moreover, the map is a homomorphism because $(a, c; *)(x, z; *) = (a + x, c + z; *)$ and this produces the desired isomorphism. □

5. Let $K \subset H$ be two subgroups of a group G. Show that $[G : K] = [G : H][H : K]$.

   Proof. If $|G| < \infty$ use $[G : K] = |G|/|K|$. If $[G : K] = \infty$ then both sides are ∞. Assume $[G : K] < \infty$. Thus $G = \sqcup g_i H$ and $H = \sqcup h_j K$. Thus $G = \sqcup g_i(\sqcup h_j K) = \sqcup g_i h_j K$ and the result follows from counting cosets. □