1. (a) Show that \( \text{Aut}(\mathbb{Q}) \cong \mathbb{Q}^\times \).  
(b) Show that \( \text{Aut}(\mathbb{R}) \supseteq \mathbb{R}^\times \). [Hint: Take a suitable \( \mathbb{Q} \)-vector space projection from \( \mathbb{R} \) to \( \mathbb{Q} \).]  
(c) (Extra credit) Find all groups \( G \) such that \( \text{Aut}(G) = \{ \text{id} \} \). [This is a fun exercise.]

2. Let \( p \) be a prime number. Consider \( G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in (\mathbb{Z}/p\mathbb{Z})^\times, b \in \mathbb{Z}/p\mathbb{Z} \right\} \).

   (a) Show that \( G \) is a group.

   (b) Let \( a \in (\mathbb{Z}/p\mathbb{Z})^\times \) and define \( H_a = \left\{ \begin{pmatrix} a^k & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z}/p\mathbb{Z}, k \in \mathbb{Z} \right\} \). Show that \( H_a \) is a normal subgroup of \( G \).

   (c) Show that every proper normal subgroup of \( G \) is of the form \( H_a \) for some \( a \). [Hint: You will need to use that \((\mathbb{Z}/p\mathbb{Z})^\times\) is a cyclic group.]

   (d) Show that \( G \cong \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times \) given by the identity map \((\mathbb{Z}/p\mathbb{Z})^\times \to \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times\).

   We'll study this group later as the Galois group of the polynomial \( X^p - 2 \).

3. Let \( G \) be a finite group and let \( H \) be a subgroup of \( G \). Denote by \( S_H \) the group of permutations of the finite set \( G/H \).

   (a) Show that if \( g \in H \) then the map \( f_g : G/H \to G/H \) defined by \( f_g(xH) = gxH \) is an element of \( S_H \).

   (b) Show that \( G \to S_H \) given by \( g \to f_g \) is a group homomorphism with kernel \( \ker f \) contained in \( H \).

   (c) Suppose that \( |G : H| = p \) is the smallest prime divisor of \( |G| \). Show that \( |G/\ker f| = p \) and deduce that \( H \) is normal in \( G \). [This is a generalization of the standard result that every index 2 subgroup is normal.]

4. Let \( G \) be an abelian group. Suppose \( g, h \in G \) have finite orders \( m \) and \( n \). Show that \( (gh) \mid [m, n] \), the least common multiple of \( m \) and \( n \).

5. Let \( G \) be a group such that \( G/Z(G) \) is cyclic. Show that \( G \) is abelian. Does the same conclusion hold if \( G/Z(G) \) is only assumed to be abelian?