Graduate Algebra
Homework 3

Fall 2014

Due 2014-09-17 at the beginning of class

1. (a) Show that $\mathrm{Aut}(\mathbb{Q}) \cong \mathbb{Q}^\times$.
   (b) Show that $\mathrm{Aut}(\mathbb{R}) \supsetneq \mathbb{R}^\times$. [Hint: Take a suitable $\mathbb{Q}$-vector space projection from $\mathbb{R}$ to $\mathbb{Q}$.]
   (c) (Extra credit) Find all groups G such that $\mathrm{Aut}(G) = \{\mathrm{id}\}$. [This is a fun exercise.]

2. Let p be a prime number. Consider $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \in (\mathbb{Z}/p\mathbb{Z})^\times, b \in \mathbb{Z}/p\mathbb{Z}\}$.
   (a) Show that G is a group.
   (b) Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and define $H_a = \{\begin{pmatrix} a^k & b \\ 0 & 1 \end{pmatrix} | b \in \mathbb{Z}/p\mathbb{Z}, k \in \mathbb{Z}\}$. Show that H_a is a normal subgroup of G.
   (c) Show that every proper normal subgroup of G is of the form H_a for some a. [Hint: You will need to use that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group.]
   (d) Show that $G \cong \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^\times$ given by the identity map $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$.

   We'll study this group later as the **Galois group** of the polynomial $X^p - 2$.

3. Let G be a finite group and let H be a subgroup of G. Denote by S_H the group of permutations of the finite set G/H.
   (a) Show that if $g \in H$ then the map $f_g : G/H \to G/H$ defined by $f_g(xH) = gxH$ is an element of S_H.
   (b) Show that $G \to S_H$ given by $g \mapsto f_g$ is a group homomorphism with kernel $\ker f$ contained in H.
   (c) Suppose that $[G : H] = p$ is the smallest prime divisor of $|G|$. Show that $|G/\ker f| = p$ and deduce that H is normal in G. [This is a generalization of the standard result that every index 2 subgroup is normal.]

4. Let G be an abelian group. Suppose $g, h \in G$ have finite orders m and n. Show that $(gh) \mid [m, n]$, the least common multiple of m and n.

5. Let G be a group such that $G/Z(G)$ is cyclic. Show that G is abelian. Does the same conclusion hold if $G/Z(G)$ is only assumed to be abelian?