1. Recall the quaternion group from homework 2.
   (a) Show that $Q$ has the following presentation: $Q \cong \langle i, j | i^2 = j^2 = (ij)^2 \rangle$.
   (b) Deduce that $|\text{Aut}(Q)| = 24$. [Hint: Make a list of the orders of the elements of $Q$.]

2. Let $G$ be the group $\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle \subset \text{GL}(2, \mathbb{R})$. Show that the subgroup of matrices with 1-s on the diagonal is not finitely generated.

3. Suppose $A$ and $B$ are two groups and $f : Z_1 \to Z_2$ is an isomorphism between the subgroups $Z_1 \subset Z(A)$ and $Z_2 \subset Z(B)$.
   (a) Show that $Z = \{(x, f(x)^{-1}) \in A \times B | x \in Z\}$ is a normal subgroup of $A \times B$. The quotient $A \ast_f B = A \times B/Z$ is called the central product of $A$ and $B$ with respect to $f$.
   (b) Show that $A \to A \ast_f B$ given by $a \mapsto (a, 1)Z$ and $B \to A \ast_f B$ given by $b \mapsto (1, b)Z$ are injective homomorphisms giving $A \cap B$ as a subgroup of $A \ast_f B$ isomorphic to $Z$.
   (c) Let $H$ be the Heisenberg group from homework 2. Consider the identity map $Z(H) \to Z(H)$. Show that the central product $H \ast_{\text{id}} H$ is isomorphic to the group of matrices

   \[
   \begin{pmatrix} 1 & a_1 & a_2 & b \\ 1 & 1 & c_1 & c_2 \\ 1 & 1 & c_2 & 1 \\
   \end{pmatrix}
   \]

   where $a_1, a_2, b, c_1, c_2 \in \mathbb{Z}/p\mathbb{Z}$

   This central product is again a Heisenberg group (think position and momentum of two particles).

4. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ define, if possible, $g \cdot z = \frac{az + b}{cz + d}$.
   (a) Show that $\text{Im}(g \cdot z) = \frac{\det(g) \text{Im}(z)}{|cz + d|^2}$.
   (b) Show that the subgroup $\text{GL}(2, \mathbb{R})^+$ of matrices with positive determinant acts on $\mathcal{H} = \{z \in \mathbb{C} | \text{Im} z > 0 \}$ via $g \mapsto (z \mapsto g \cdot z)$.
   (c) Show that this action is transitive, i.e., all of $\mathcal{H}$ is one orbit, and compute $\text{Stab}(i)$ and $\text{Stab}(\zeta_3)$.

5. Let $\text{GL}(2, \mathbb{Z})$ consist of $2 \times 2$ matrices with entries in $\mathbb{Z}$ and determinant $\pm 1$.
   (a) Show that $\text{GL}(2, \mathbb{Z})$ is a group and that it acts on $\mathbb{Z}^2$ by matrix multiplication.
   (b) Show that the set $S = \{ \begin{pmatrix} d \\ 0 \end{pmatrix} | d \in \mathbb{Z}_{\geq 1} \}$ parametrizes the orbits of $\text{GL}(2, \mathbb{Z})$ acting on $\mathbb{Z}^2$, i.e., in each orbit there is a unique element from the set $S$ and this provides a bijection between the orbits and the set $S$. [Hint: Show that the orbit of $\begin{pmatrix} d \\ 0 \end{pmatrix}$ consists of $\begin{pmatrix} a \\ b \end{pmatrix}$ with $(a, b) = d$.]