Graduate Algebra
Homework 4

Fall 2014

Due 2014-09-24 at the beginning of class

1. Recall the quaternion group from homework 2.
   (a) Show that Q has the following presentation: $Q \cong \langle i, j | i^2 = j^2 = (ij)^2 \rangle$.
   (b) Deduce that $|\operatorname{Aut}(Q)| = 24$. [Hint: Make a list of the orders of the elements of Q.]

2. Let G be the group $\langle \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \rangle \subset \mathrm{GL}(2, \mathbb{R})$. Show that the subgroup of matrices with 1-s on the diagonal is not finitely generated.

3. Suppose A and B are two groups and $f : Z_1 \to Z_2$ is an isomorphism between the subgroups $Z_1 \subset Z(A)$ and $Z_2 \subset Z(B)$.
   (a) Show that $Z = \{(x, f(x)^{-1}) \in A \times B | x \in Z\}$ is a normal subgroup of $A \times B$. The quotient $A *_f B = A \times B/Z$ is called the **central product** of A and B with respect to f.
   (b) Show that $A \to A *_f B$ given by $a \mapsto (a, 1)Z$ and $B \to A *_f B$ given by $b \mapsto (1, b)Z$ are injective homomorphisms giving $A \cap B$ as a subgroup of $A *_f B$ isomorphic to Z.
   (c) Let H be the Heisenberg group from homework 2. Consider the identity map $Z(H) \to Z(H)$. Show that the central product $H *_{\mathrm{id}} H$ is isomorphic to the group of matrices

$$\{ \begin{pmatrix} 1 & a_1 & a_2 & b \\ & 1 & & c_1 \\ & & 1 & c_2 \\ & & & 1 \end{pmatrix} | a_1, a_2, b, c_1, c_2 \in \mathbb{Z}/p\mathbb{Z} \}$$

   This central product is again a Heisenberg group (think position and momentum of two particles).

4. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ define, if possible, $g \cdot z = \frac{az+b}{cz+d}$.
   (a) Show that $\operatorname{Im}(g \cdot z) = \frac{\det(g) \operatorname{Im}(z)}{|cz+d|^2}$.
   (b) Show that the subgroup $\mathrm{GL}(2, \mathbb{R})^+$ of matrices with positive determinant acts on $\mathcal{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ via $g \mapsto (z \mapsto g \cdot z)$.
   (c) Show that this action is transitive, i.e., all of $\mathcal{H}$ is one orbit, and compute $\operatorname{Stab}(i)$ and $\operatorname{Stab}(\zeta_3)$.

5. Let $\mathrm{GL}(2, \mathbb{Z})$ consist of 2×2 matrices with entries in $\mathbb{Z}$ and determinant ± 1.
   (a) Show that $\mathrm{GL}(2, \mathbb{Z})$ is a group and that it acts on $\mathbb{Z}^2$ by matrix multiplication.
   (b) Show that the set $S = \{ \begin{pmatrix} d \\ 0 \end{pmatrix} | d \in \mathbb{Z}_{\geq 1} \}$ parametrizes the orbits of $\mathrm{GL}(2, \mathbb{Z})$ acting on $\mathbb{Z}^2$, i.e., in each orbit there is a unique element from the set S and this provides a bijection between the orbits and the set S. [Hint: Show that the orbit of $\begin{pmatrix} d \\ 0 \end{pmatrix}$ consists of $\begin{pmatrix} a \\ b \end{pmatrix}$ with $(a, b) = d$.]