Graduate Algebra
Homework 5

Fall 2014

Due 2014-10-01 at the beginning of class

1. Let $n \geq 5$.
   (a) Show that the only proper normal subgroup of S_n is A_n.
   (b) Let H be a subgroup of S_n. Show that either $H = A_n$ or $[S_n : H] \geq n$. [Hint: Consider the action of S_n on S_n/H.]
2. Let G be a finite group and N the intersection of all p-Sylow subgroups of G. Show that N is a normal p-subgroup of G and that every normal p-subgroup of G is contained in N.
3. Let $2 < p < q$ be two primes such that $p \mid q + 1$. Let G be a group with $|G| = p^2q^2$.
   (a) Show that there is a normal q-Sylow subgroup Q of G. [Hint: Show that $q \nmid p^2 - 1$.]
   (b) Let P be a p-Sylow subgroup. Show that $G \cong Q \rtimes P$.
   (c) If Q is cyclic show that G is abelian.
   (d) List all isomorphism classes of abelian groups of order p^2q^2 with $p \neq q$.

   There are nonabelian G of the form $(\mathbb{Z}/q\mathbb{Z})^2 \rtimes (\mathbb{Z}/p\mathbb{Z})^2$, at least two nonisomorphic such semidirect products. Cf. `http://www.icm.tu-bs.de/ag_algebra/software/small/number.html`
4. Let G be a finite group of order 231.
   (a) Show that G has normal 7-Sylow and 11-Sylow subgroups.
   (b) Show that for groups A, B, C, $(A \rtimes_f B) \times C \cong (A \times C) \rtimes_{f \times \mathrm{id}} B$ where $f \times \mathrm{id} : B \to \mathrm{Aut}(A) \times \mathrm{Aut}(C) \subset \mathrm{Aut}(A \times C)$ sends everything to the trivial automorphism of C.
   (c) Show that the unique 11-Sylow subgroup of G is contained in $Z(G)$. [Hint: Use part (b) to express the 11-Sylow subgroup as a direct factor of G.]
5. Let $\mathbb{F}_q$ be a finite field with q elements and V an n-dimensional vector space over $\mathbb{F}_q$.
   (a) Show that $\mathrm{GL}(n, \mathbb{F}_q)$, the group of $n \times n$ matrices with coefficients in $\mathbb{F}_q$ and nonzero determinant, acts **simply transitively** on the set of all possible bases of V. Here transitive means that there is one single orbit (for any x, y there exists g such that $gx = y$) and simple means that if $gx = x$ for some x then $g = 1$.
   (b) Deduce that
   $$|\mathrm{GL}(n, \mathbb{F}_q)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$$

   This formula is useful in random algorithms where it computes the probability that a random matrix is invertible.