Graduate Algebra
Homework 6

Fall 2014

Due 2014-10-08 at the beginning of class

1. Show that $S_n^{\text{ab}} \cong \mathbb{Z}/2\mathbb{Z}$ by showing that $[S_n, S_n] = A_n$.
2. Suppose G is a finite group with p^3 elements where $p > 2$ is odd.
   (a) Find all possibilities for G abelian.
   (b) For the rest of the problem suppose G is not abelian. Show that $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$.
   (c) Conclude that $[G, G] = Z(G)$. [Hint: Use the universal property of abelianization.]
   (d) Suppose G has an element a of order p^2 and suppose that every $b \notin \langle a \rangle$ also has order p^2.
      i. Show that $b^p = a^{pk}$ for some k coprime to p.
      ii. Verify by induction that $(a^k b^{-1})^n = a^{kn} b^{-n} [b, a^{-k}]^{n(n-1)/2}$ for all n. [Hint: Use (c).]
      iii. Conclude that $a^k b^{-1}$ has order p and is not in $\langle a \rangle$, thus getting a contradiction.
   (e) Suppose G has an element a of order p^2. Show that $G \cong \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.
3. Let G be a finite group, H a normal subgroup and $P \in \mathrm{Syl}_p(H)$.
   (a) Show that $gPg^{-1} \in \mathrm{Syl}_p(H)$ for every $g \in G$. [Here $g \in G$ not only in H.]
   (b) Deduce that there exists $h \in H$ such that $h^{-1}g \in N_G(P)$.
   (c) Show that $G = HN_G(P)$.
   (d) Deduce that $[G : H] \mid |N_G(P)|$.
4. Let G be a group. A subgroup H is said to be **maximal** if it is not contained properly in any proper subgroup of G.
   (a) Show that if G is finite then every proper subgroup of G is contained in a maximal subgroup of G.
   (b) What are the maximal subgroups of $\mathbb{Z}$?
   (c) Show that $\mathbb{Q}$ has no maximal subgroups.
5. For a finite group G let $\Phi(G)$ be the intersection of all maximal subgroups of G (if no proper subgroup exists, define $\Phi(G) = G$).
   (a) Show that $\Phi(G) \lhd G$.
   (b) Show that every Sylow subgroup of $\Phi(G)$ is normal in G. [Hint: Use the previous two problems.]
   (c) Find $\Phi(S_n)$ and $\Phi(A_n)$ for all $n \geq 2$.

   The group $\Phi(G)$ is called the Frattini subgroup of G. One application of this problem is to Galois theory next semester, where it implies that the composite of all minimal subextensions of a Galois extension is Galois.