1. Show that $S^b_n \cong \mathbb{Z}/2\mathbb{Z}$ by showing that $[S_n, S_n] = A_n$.

   Proof. Know that $[G, G] < G$ and so $[S_n, S_n]$ is one of $1, A_n$ or $S_n$ by the last homework. It’s not 1 since $S_n$ is not abelian, and $\varepsilon(ggh^{-1}h^{-1}) = 1$ and so $[g, h] \in A_n$ for all $g, h \in S_n$ thus $[S_n, S_n] = A_n$. Thus $S^n = S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$. \qed

2. Suppose $G$ is a finite group with $p^3$ elements where $p > 2$ is odd.

   (a) Find all possibilities for $G$ abelian.

   (b) For the rest of the problem suppose $G$ is not abelian. Show that $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$.

   (c) Conclude that $[G, G] = Z(G)$. [Hint: Use the universal property of abelianization.]

   (d) Suppose $G$ has an element $a$ of order $p^2$ and suppose that every $b \notin \langle a \rangle$ also has order $p^2$.

      i. Show that $b^p = a^{pk}$ for some $k$ coprime to $p$.

      ii. Verify by induction that $(a^{kb^{-1}})^n = a^{knb^{-n}[b, a^{-k}]n(n-1)/2}$ for all $n$. [Hint: Use (c).]

      iii. Conclude that $a^{kb^{-1}}$ has order $p$ and is not in $\langle a \rangle$, thus getting a contradiction.

   (e) Suppose $G$ has an element $a$ of order $p^2$. Show that $G \cong \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

   Proof. (a) Write $|G| = n_1 \cdots n_r$ with $n_i | \cdots | n_1$. Here $p^3 = p^3 = p^2 \cdot p = p \cdot p \cdot p$. Thus $\mathbb{Z}/p^3\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $(\mathbb{Z}/p\mathbb{Z})^3$ are the only abelian groups of order $p^3$.

   (b): Since $G$ is not abelian, homework 3 problem 5 gives that $G/Z(G)$ is not cyclic. Moreover, $Z(G)$ is nontrivial since $G$ is a $p$-group and so the only possibility is $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$ with $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$.

   (c): The projection map $G \to G/Z(G)$ is induced from $G^{ab} = G/[G, G] \to G/Z(G)$ by the universal property of abelianization, since $G/Z(G)$ is abelian. Thus $[G, G]$ is contained in the kernel of this map, which is $Z(G)$. But $[G, G] \neq 1$ since $G$ is not abelian and there is only one nontrivial subgroup of $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$, namely $Z(G)$ itself.

   (d):

   (i): $b^p$ has order $p$ so is in $\langle a \rangle$ and thus is of the form $a^{pk}$ since these are the only order $p$ powers of $a$.

   (ii): Base case is $n = 1$, contentless. We keep using that $[G, G] = Z(G)$ so all commutators commute with everything. For the inductive hypothesis need to check that

   \[
   a^{k(n+1)b^{-n-1}}[b, a^{-k}]n(n+1)/2 = (a^{kb^{-1}})^n+1
   \]

   \[
   = (a^{kb^{-1}})^na^{kb^{-1}}
   \]

   \[
   = a^{knb^{-n}}[b, a^{-k}]n(n-1)/2a^{kb^{-1}}
   \]
which is equivalent to
\[
\begin{align*}
  a^k b^{-n} [b, a^{-k}]^n &= b^{-n} a^k \\
  [b, a^{-k}]^n &= b^n a^{-k} b^{-n} a^k
\end{align*}
\]

This again check by induction. The base case is again contentless. For the inductive hypothesis:
\[
\begin{align*}
  b^{n+1} a^{-k} b^{-n-1} a^k &= b(b^n a^{-k} b^{-n}) h^{-1} a^k \\
  &= b([b, a^{-k}]^n a^{-k}) h^{-1} a^k \\
  &= [b, a^{-k}]^{n+1}
\end{align*}
\]

(iii): Thus \((a^k b^{-1})^p = a^{pk} b^{-p} [b, a^{-k}]^{p(p-1)/2} = 1\) since \(a^{pk} = b^p\) and \(|[G, G]| = p\); here we used that \(p(p-1)/2\) is divisible by \(p\) if \(p\) is odd. Also \(a^k b^{-1} \notin \langle a \rangle\) since \(b\) is not a power of \(a\) and we get our contradiction.

(e): From (d) there exists \(b \notin \langle a \rangle\) of order \(p\). Then \(N = \langle a \rangle \triangleleft G\) because it has index \(p\) the smallest prime divisor of \(|G|\) and this intersect trivially with \(H = \langle b \rangle\). Thus \(NH\) is a subgroup of \(G\) and has order \(p^2 \cdot p = |G|\) so \(G = NH\). Thus \(G \cong N \times H\) as desired.

\[
\square
\]

3. Let \(G\) be a finite group, \(H\) a normal subgroup and \(P \in \text{Syl}_p(H)\).

(a) Show that \(gPg^{-1} \in \text{Syl}_p(H)\) for every \(g \in G\). [Here \(g \in G\) not only in \(H\].]

(b) Deduce that there exists \(h \in H\) such that \(h^{-1} g \in N_G(P)\).

(c) Show that \(G = HN_G(P)\).

(d) Deduce that \(|G : H| = |N_G(P)|\).

Proof. (a): \(gPg^{-1} \subset gHg^{-1} = H\) by normality and has same order as \(P\) thus \(gPg^{-1} \in \text{Syl}_p(H)\).

(b): All \(p\)-Sylow are conjugate in \(H\) so there exists \(h \in H\) with \(gPg^{-1} = hPh^{-1}\). Thus \(h^{-1} g P (h^{-1} g)^{-1} = P\) so \(h^{-1} g \in N_G(P)\).

(c): Since \(H\) is normal, \(HN_G(P)\) is a subgroup of \(G\) and since \(h^{-1} g \in N_G(P)\) we deduce that \(G = HN_G(P)\).

(d): From the isomorphism theorems \(|G/H| = |HN_G(P)/H| = |N_G(P)/N_G(P) \cap H| = |N_G(P)|\) as desired.

\[
\square
\]

4. Let \(G\) be a group. A subgroup \(H\) is said to be **maximal** if it is not contained properly in any proper subgroup of \(G\).

(a) Show that if \(G\) is finite then every proper subgroup of \(G\) is contained in a maximal subgroup of \(G\).

(b) What are the maximal subgroups of \(\mathbb{Z}\)?

(c) Show that \(\mathbb{Q}\) has no maximal subgroups.

Proof. (a): By induction on \(|G : H|\). If there are no subgroups between \(H\) and \(G\) then \(H\) by definition is maximal. This is the base case. For the inductive step: if \(H \subsetneq K \subsetneq G\) for a subgroup \(K\), then \(|G : K| < |G : H|\) and by induction \(K\) and thus also \(H\) is in some maximal subgroup of \(G\).

(b): We already know that the subgroups of \(\mathbb{Z}\) are of the form \(n\mathbb{Z}\). Note that \(n\mathbb{Z} \subset m\mathbb{Z}\) iff \(m | n\) and so the maximal subgroups are precisely \(p\mathbb{Z}\) for \(p\) prime.
(c): Suppose $G$ is a maximal subgroup of $\mathbb{Q}$. Pick $p/q \in \mathbb{Q} - G$. As in the case of $\mathbb{Z}$, the subgroups of $(1/qn)\mathbb{Z}$ are all of the form $(l/qn)\mathbb{Z}$ for some $l \in \mathbb{Z}$. Thus $G \cap (1/qn)\mathbb{Z} = (l_n/qn)\mathbb{Z}$ for some $l_n \in \mathbb{Z}$.

Look at $\frac{1}{qnl_n} \in \mathbb{Q} = p/q\mathbb{Z} + G$. We deduce that there exists $m \in \mathbb{Z}$ such that $1/(qnl_n) = pm/q + g$ for some $g \in G$ which is then $g = \frac{1 - pmnl_n}{qnl_n}$. But then $ng = \frac{1 - pmnl_n}{qn} \in G \cap \frac{1}{qn} \mathbb{Z} = \frac{l_n}{qn} \mathbb{Z}$ which implies that $l_n \mid 1 - pmnl_n$ and so $l_n = 1$.

Thus $G \cap \frac{1}{qn} \mathbb{Z} = \frac{1}{qn} \mathbb{Z}$ and so $G = \mathbb{Q}$ since every rational number is in some $\frac{1}{qn} \mathbb{Z}$.

\hfill \Box

5. For a finite group $G$ let $\Phi(G)$ be the intersection of all maximal subgroups of $G$ (if no proper subgroup exists, define $\Phi(G) = G$).

(a) Show that $\Phi(G) \triangleleft G$.

(b) Show that every Sylow subgroup of $\Phi(G)$ is normal in $G$. [Hint: Use the previous two problems.]

(c) Find $\Phi(S_n)$ and $\Phi(A_n)$ for all $n \geq 2$.

The group $\Phi(G)$ is called the Frattini subgroup of $G$. One application of this problem is to Galois theory next semester, where it implies that the composite of all minimal subextensions of a Galois extension is Galois.

Proof. (a): If $H$ is maximal in $G$ then $gHg^{-1}$ is also maximal because $gHg^{-1} \subset K$ for $K$ proper implies $H \subset g^{-1}Kg$. Thus $g\Phi(G)g^{-1} = \cap gHg^{-1} = \cap H = \Phi(G)$ where $H$ runs over the maximal subgroups of $G$.

(b): Let $P$ be a Sylow subgroup of $\Phi(G) \triangleleft G$. Then problem 3 implies that $G = \Phi(G)N_G(P)$. If $N_G(P) = G$ then $P \triangleleft G$, by definition. If not then $N_G(P)$ is contained in some maximal subgroup $H$ of $G$ by part (a). But also $\Phi(G) \subset H$ and so $\Phi(G)N_G(P) \subset H$ which contradicts that the composite is $G$.

(c): $\Phi(A_n) \triangleleft A_n$ and $\Phi(S_n) \triangleleft S_n$.

Suppose $n \geq 5$. Then $\Phi(A_n) = 1$ or $A_n$ and $\Phi(S_n)$ is one of 1, $A_n$ and $S_n$. As $A_n$ and $S_n$ have proper Sylow subgroups which cannot be normal in $A_n$ respectively $S_n$ it follows that $\Phi(A_n) = 1$ and $\Phi(S_n) = 1$.

Suppose $n = 2, 3$. Then $\Phi(A_n) = A_n$ by definition since $A_n$ has prime order. $\Phi(S_2) = S_2$ by definition and $\Phi(S_3) = 1$ as $A_3$ and $\langle (12) \rangle$ are maximal subgroups.

Suppose $n = 4$. Again $\Phi(A_4) \neq A_4$ and is a normal subgroup of $A_4$. The group $A_4$ has no index 2 subgroup as such a group would be normal and would have a unique 3-Sylow subgroup. But then by the lemma from class $A_4$ would have a unique 3-Sylow subgroup which is not the case. Thus the unique 2-Sylow subgroup of $A_4$ and any of the 3-Sylow subgroups are in fact maximal and so $\Phi(A_4) = 1$.

What about $\Phi(S_4)$? Certainly $A_4$ is maximal and from the homework every maximal subgroup $H \neq A_4$ would have to have $[G : H] \geq 4$. Thus $S_{\{i,j,k\}} \subset S_4$ is also maximal for $i, j, k$ distinct among 1, 2, 3, 4. This implies that $\Phi(S_4) \subset A_4 \cap (\cap S_{\{i,j,k\}}) = 1$.