Graduate Algebra
Homework 6

Fall 2014

Due 2014-10-08 at the beginning of class

1. Show that $S_n^{\text{ab}} \cong \mathbb{Z}/2\mathbb{Z}$ by showing that $[S_n, S_n] = A_n$.

   Proof. Know that $[G,G] \lhd G$ and so $[S_n, S_n]$ is one of $1, A_n$ or S_n by the last homework. It's not 1 since S_n is not abelian, and $\varepsilon(ghg^{-1}h^{-1}) = 1$ and so $[g,h] \in A_n$ for all $g, h \in S_n$ thus $[S_n, S_n] = A_n$. Thus $S_n^{\text{ab}} = S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

2. Suppose G is a finite group with p^3 elements where $p > 2$ is odd.

   (a) Find all possibilities for G abelian.

   (b) For the rest of the problem suppose G is not abelian. Show that $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$.

   (c) Conclude that $[G,G] = Z(G)$. [Hint: Use the universal property of abelianization.]

   (d) Suppose G has an element a of order p^2 and suppose that every $b \notin \langle a \rangle$ also has order p^2.

      i. Show that $b^p = a^{pk}$ for some k coprime to p.

      ii. Verify by induction that $(a^k b^{-1})^n = a^{kn} b^{-n} [b, a^{-k}]^{n(n-1)/2}$ for all n. [Hint: Use (c).]

      iii. Conclude that $a^k b^{-1}$ has order p and is not in $\langle a \rangle$, thus getting a contradiction.

   (e) Suppose G has an element a of order p^2. Show that $G \cong \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.

   Proof. (a) Write $|G| = n_1 \cdots n_r$ with $n_r \mid \ldots \mid n_1$. Here $p^3 = p^3 = p^2 \cdot p = p \cdot p \cdot p$. Thus $\mathbb{Z}/p^3\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $(\mathbb{Z}/p\mathbb{Z})^3$ are the only abelian groups of order p^3.

   (b): Since G is not abelian, homework 3 problem 5 gives that $G/Z(G)$ is not cyclic. Moreover, $Z(G)$ is nontrivial since G is a p-group and so the only possibility is $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$ with $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$.

   (c): The projection map $G \to G/Z(G)$ is induced from $G^{\text{ab}} = G/[G,G] \to G/Z(G)$ by the universal property of abelianization, since $G/Z(G)$ is abelian. Thus $[G,G]$ is contained in the kernel of this map, which is $Z(G)$. But $[G,G] \neq 1$ since G is not abelian and there is only one nontrivial subgroup of $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$, namely $Z(G)$ itself.

   (d):

   (i): b^p has order p so is in $\langle a \rangle$ and thus is of the form a^{pk} since these are the only order p powers of a.

   (ii): Base case is $n = 1$, contentless. We keep using that $[G,G] = Z(G)$ so all commutators commute with everything. For the inductive hypothesis need to check that

$$\begin{aligned} a^{k(n+1)} b^{-n-1} [b, a^{-k}]^{n(n+1)/2} &= (a^k b^{-1})^{n+1} \\ &= (a^k b^{-1})^n a^k b^{-1} \\ &= a^{kn} b^{-n} [b, a^{-k}]^{n(n-1)/2} a^k b^{-1} \end{aligned}$$

which is equivalent to

$$\begin{aligned} a^k b^{-n}[b, a^{-k}]^n &= b^{-n} a^k \\ [b, a^{-k}]^n &= b^n a^{-k} b^{-n} a^k \end{aligned}$$

This again check by induction. The base case is again contentless. For the inductive hypothesis:

$$\begin{aligned} b^{n+1} a^{-k} b^{-n-1} a^k &= b(b^n a^{-k} b^{-n}) b^{-1} a^k \\ &= b([b, a^{-k}]^n a^{-k}) b^{-1} a^k \\ &= [b, a^{-k}]^{n+1} \end{aligned}$$

(iii): Thus $(a^k b^{-1})^p = a^{pk} b^{-p} [b, a^{-k}]^{p(p-1)/2} = 1$ since $a^{pk} = b^p$ and $|[G, G]| = p$; here we used that $p(p-1)/2$ is divisible by p if p is odd. Also $a^k b^{-1} \notin \langle a \rangle$ since b is not a power of a and we get our contradiction.

(e): From (d) there exists $b \notin \langle a \rangle$ of order p. Then $N = \langle a \rangle \lhd G$ because it has index p the smallest prime divisor of $|G|$ and this intersect trivially with $H = \langle b \rangle$. Thus NH is a subgroup of G and has order $p^2 \cdot p = |G|$ so $G = NH$. Thus $G \cong N \rtimes H$ as desired.

□

3. Let G be a finite group, H a normal subgroup and $P \in \mathrm{Syl}_p(H)$.

   (a) Show that $gPg^{-1} \in \mathrm{Syl}_p(H)$ for every $g \in G$. [Here $g \in G$ not only in H.]

   (b) Deduce that there exists $h \in H$ such that $h^{-1}g \in N_G(P)$.

   (c) Show that $G = HN_G(P)$.

   (d) Deduce that $[G : H] \mid |N_G(P)|$.

Proof. (a): $gPg^{-1} \subset gHg^{-1} = H$ by normality and has same order as P thus $gPg^{-1} \in \mathrm{Syl}_p(H)$.

(b): All p-Sylow are conjugate in H so there exists $h \in H$ with $gPg^{-1} = hPh^{-1}$. Thus $h^{-1}gP(h^{-1}g)^{-1} = P$ so $h^{-1}g \in N_G(P)$.

(c): Since H is normal, $HN_G(P)$ is a subgroup of G and since $h^{-1}g \in N_G(P)$ we deduce that $G = HN_G(P)$.

(d): From the isomorphism theorems $|G/H| = |HN_G(P)/H| = |N_G(P)/N_G(P) \cap H| \mid |N_G(P)|$ as desired. □

4. Let G be a group. A subgroup H is said to be **maximal** if it is not contained properly in any proper subgroup of G.

   (a) Show that if G is finite then every proper subgroup of G is contained in a maximal subgroup of G.

   (b) What are the maximal subgroups of $\mathbb{Z}$?

   (c) Show that $\mathbb{Q}$ has no maximal subgroups.

Proof. (a): By induction on $[G : H]$. If there are no subgroups between H and G then H by definition is maximal. This is the base case. For the inductive step: if $H \subsetneq K \subsetneq G$ for a subgroup K, then $[G : K] < [G : H]$ and by induction K and thus also H is in some maximal subgroup of G.

(b): We already know that the subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$. Note that $n\mathbb{Z} \subset m\mathbb{Z}$ iff $m \mid n$ and so the maximal subgroups are precisely $p\mathbb{Z}$ for p prime.

(c): Suppose G is a maximal subgroup of $\mathbb{Q}$. Pick $p/q \in \mathbb{Q} - G$. As in the case of $\mathbb{Z}$, the subgroups of $(1/qn)\mathbb{Z}$ are all of the form $(l/qn)\mathbb{Z}$ for some $l \in \mathbb{Z}$. Thus $G \cap (1/qn)\mathbb{Z} = (l_n/qn)\mathbb{Z}$ for some $l_n \in \mathbb{Z}$. Look at $\dfrac{1}{qnl_n} \in \mathbb{Q} = p/q\mathbb{Z} + G$. We deduce that there exists $m \in \mathbb{Z}$ such that $1/(qnl_n) = pm/q + g$ for some $g \in G$ which is then $g = \dfrac{1 - pmnl_n}{qnl_n}$. But then $ng = \frac{1-pmnl_n}{qn} \in G \cap \frac{1}{qn}\mathbb{Z} = \frac{l_n}{qn}\mathbb{Z}$ which implies that $l_n \mid 1 - pmnl_n$ and so $l_n = 1$.

Thus $G \cap \frac{1}{qn}\mathbb{Z} = \frac{1}{qn}\mathbb{Z}$ and so $G = \mathbb{Q}$ since every rational number is in some $\frac{1}{qn}\mathbb{Z}$.

$\square$

5. For a finite group G let $\Phi(G)$ be the intersection of all maximal subgroups of G (if no proper subgroup exists, define $\Phi(G) = G$).

   (a) Show that $\Phi(G) \lhd G$.

   (b) Show that every Sylow subgroup of $\Phi(G)$ is normal in G. [Hint: Use the previous two problems.]

   (c) Find $\Phi(S_n)$ and $\Phi(A_n)$ for all $n \geq 2$.

   The group $\Phi(G)$ is called the Frattini subgroup of G. One application of this problem is to Galois theory next semester, where it implies that the composite of all minimal subextensions of a Galois extension is Galois.

Proof. (a): If H is maximal in G then gHg^{-1} is also maximal because $gHg^{-1} \subset K$ for K proper implies $H \subset g^{-1}Kg$. Thus $g\Phi(G)g^{-1} = \cap gHg^{-1} = \cap H = \Phi(G)$ where H runs over the maximal subgroups of G.

(b): Let P be a Sylow subgroup of $\Phi(G) \lhd G$. Then problem 3 implies that $G = \Phi(G)N_G(P)$. If $N_G(P) = G$ then $P \lhd G$, by definition. If not then $N_G(P)$ is contained in some maximal subgroup H of G by part (a). But also $\Phi(G) \subset H$ and so $\Phi(G)N_G(P) \subset H$ which contradicts that the composite is G.

(c): $\Phi(A_n) \lhd A_n$ and $\Phi(S_n) \lhd S_n$.

Suppose $n \geq 5$. Then $\Phi(A_n)$ is 1 or A_n and $\Phi(S_n)$ is one of 1, A_n and S_n. As A_n and S_n have proper Sylow subgroups which cannot be normal in A_n respectively S_n it follows that $\Phi(A_n) = 1$ and $\Phi(S_n) = 1$.

Suppose $n = 2, 3$. Then $\Phi(A_n) = A_n$ by definition since A_n has prime order. $\Phi(S_2) = S_2$ by definition and $\Phi(S_3) = 1$ as A_3 and $\langle(12)\rangle$ are maximal subgroups.

Suppose $n = 4$. Again $\Phi(A_4) \neq A_4$ and is a normal subgroup of A_4. The group A_4 has no index 2 subgroup as such a group would be normal and would have a unique 3-Sylow subgroup. But then by the lemma from class A_4 would have a unique 3-Sylow subgroup which is not the case. Thus the unique 2-Sylow subgroup of A_4 and any of the 3-Sylow subgroups are in fact maximal and so $\Phi(A_4) = 1$.

What about $\Phi(S_4)$? Certainly A_4 is maximal and from the homework every maximal subgroup $H \neq A_4$ would have to have $[G : H] \geq 4$. Thus $S_{\{i,j,k\}} \subset S_4$ is also maximal for i, j, k distinct among $1, 2, 3, 4$. This implies that $\Phi(S_4) \subset A_4 \cap (\cap S_{\{i,j,k\}}) = 1$. $\square$