Graduate Algebra
Homework 7

Fall 2014

Due 2014-10-29 at the beginning of class

1. Let N be a normal subgroup of G. If N and G/N are solvable, show that G is solvable.
2. Show that D_{2n} is nilpotent iff n is a power of 2.
3. Let $I = \mathbb{Z}_{n\geq 1}$ with partial order $m \leq n$ iff $m \mid n$.
   (a) Show that I is a directed set.
   (b) Let $G_n = \mathbb{Z}$ and for $m \mid n$ let $\iota_{m,n}(x) = xn/m$. Show that (G_n) is a direct system of groups.
   (c) Show that $\varinjlim G_n \cong \mathbb{Q}$.
4. Show that $\mathbb{Z}_p$ is torsion-free, i.e., there is no element $x \in \mathbb{Z}_p$ such that $mx = 0$ for some nonzero integer m. [Here $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.]
5. Show that every open subgroup of a topological group is closed in the topology. Deduce that every open subgroup of a profinite group is compact.
6. Let $(G_u)_{u\in I}$ be a direct system of finite groups with homomorphisms $\iota_{u,v} : G_u \to G_v$ for $u \leq v$ and $\iota_u : G_u \to G := \varinjlim G_u$. If H is a subgroup of G show that $(H_u)_{u\in I}$ with $H_u = \iota_u^{-1}(H)$ is a direct system of groups with $H = \varinjlim H_u$.
7. Let G be a topological group and $\widehat{G}$ its Pontryagin dual with the dual topology.
   (a) If G is compact show that $\widehat{G}$ has the discrete topology.
   (b) Compute the Pontryagin dual of $\mathbb{R}/\mathbb{Z}$.